Problema J166. Let P be a point inside triangle ABC and let d_a, d_b, d_c be the distances from point P to the sides of the triangle. Prove that

$$\frac{K}{d_a d_b d_c} \geq \frac{s}{Rr}$$

where K is the area of the pedal triangle of P and s, R, r are the semiperimeter, circumradius and inradius of triangle ABC.

Proposed by Andrei Razvan Baleanu, "George Cosbuc", Motru, Romania

Solution by Ercole Suppa, Teramo, Italy

By using the AM-GM inequality and the well known relation $abc = 4RK$, we get

$$8K^3 = (2K)^3 = (ad_a + bd_b + cd_c)^3 \geq 27abc \cdot d_a d_b d_c = 27 \cdot 4RK \cdot d_a d_b d_c$$

from which it follows that

$$d_a d_b d_c \leq \frac{2}{27} \cdot \frac{K^2}{R} \tag{1}$$

Since the function $f(x) = \sin x$ is concave on $[0, \pi]$, the Jensen's inequality yields

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A+B+C}{3}\right) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

Thus, taking into account the extended sine law, we have

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2} \quad \Rightarrow \quad \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \leq \frac{3\sqrt{3}}{2} \quad \Rightarrow$$

$$2s \leq 3\sqrt{3}R \quad \Rightarrow \quad 4s^2 \leq 27R^2 \tag{2}$$

From (1) and (2), by using the formula $K = rs$, we deduce that

$$d_a d_b d_c \leq \frac{2}{27} \cdot \frac{r^2 s^2}{R} \leq \frac{27R^2 r^2}{54R} = \frac{Rr^2}{2} \quad \Rightarrow$$

$$\frac{K}{d_a d_b d_c} \geq \frac{2K}{Rr^2} = \frac{2K}{Rr \cdot \frac{K}{s}} = \frac{2s}{Rr} > \frac{s}{Rr}$$

which ends the proof. $\square$