Problema J168. Let n be a positive integer. Find the least positive integer a such that the system

$$\begin{cases} x_1 + x_2 + \cdots + x_n = a \\ \\ x_1^2 + x_2^2 + \cdots + x_n^2 = a \end{cases}$$

has no integer solutions.

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If $a = n + 1$ the system

$$\begin{cases} x_1 + x_2 + \cdots + x_n = n + 1 \\ \\ x_1^2 + x_2^2 + \cdots + x_n^2 = n + 1 \end{cases}$$

may not have integer solutions $(x_1, \ldots, x_n)$, otherwise there would be

$$\begin{aligned} (x_1 - 1)^2 + \cdots + (x_n - 1)^2 &= \sum x_i^2 - 2\sum x_i + n = \\ &= n + 1 - 2(n + 1) + n = -1 \end{aligned}$$

which is impossible.

If $a \leq n$ the system admit the obviuos solution $x_1 = x_2 = \cdots = x_a = 1$, $x_{a+1} = \cdots = x_n = 0$.

Therefore the least positive integer a such that the system has not integer solutions is $a = n + 1$. □