Problema J169. If $x, y, z > 0$ and $x + y + z = 1$, find the maximum of

$$E(x, y, z) = \frac{xy}{x+y} + \frac{yz}{y+z} + \frac{zx}{z+x}$$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

Solution by Ercole Suppa, Teramo, Italy

According to AM-GM inequality we have

$$\begin{aligned} E(x,y,z) &\leq \frac{\frac{(x+y)^2}{4}}{x+y} + \frac{\frac{(y+z)^2}{4}}{y+z} + \frac{\frac{(z+x)^2}{4}}{z+x} = \\ &= \frac{x+y}{4} + \frac{y+z}{4} + \frac{z+x}{4} = \\ &= \frac{x+y+z}{2} = \frac{1}{2} \end{aligned}$$

Therefore the maximum of $E(x, y, z)$ is $\frac{1}{2}$, occurring for example if

$$x = y = z = \frac{1}{3} \qquad \square$$