Problema J172. Let P be a point situated in the interior of an equilateral triangle ABC and let A', B', C' be the intersections of lines AP, BP, CP with sides BC, CA, AB, respectively. Find P such that

$$A'B^2 + B'C^2 + C'A^2 = AB'^2 + BC'^2 + CA'^2$$

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Let us denote by a the side length of equilateral triangle ABC and let $BA' = x$, $CB' = y$, $AC' = z$. Then $A'C = a - x$, $B'A = a - y$, $C'B = a - z$ and given condition yields

$$x^2 + y^2 + z^2 = (a-x)^2 + (a-y)^2 + (a-z)^2 \quad \Leftrightarrow \quad 2(x+y+z) = 3a \qquad (1)$$

Since the AA', BB', CC' are concurrent, the Ceva's theorem gives

$$\begin{aligned} &\frac{x}{a-x} \cdot \frac{y}{a-y} \cdot \frac{z}{a-z} = 1 \quad \Rightarrow \\ &xyz = (a-x)(a-y)(a-z) \end{aligned} \qquad (2)$$

From (1) and (2) we obtain

$$\begin{aligned} &a^3 - (x+y+z)a^2 + (xy+yz+zx)a - 2xyz = 0 && \Rightarrow \\ &4a^3 - 4(x+y+z)a^2 + 4(xy+yz+zx)a - 8xyz = 0 && \Rightarrow \\ &4a^3 - 3a^3 - 2(x+y+z)a^2 + 4(xy+yz+zx)a - 8xyz = 0 && \Rightarrow \\ &a^3 - 2(x+y+z)a^2 + 4(xy+yz+zx)a - 8xyz = 0 && \Rightarrow \\ &(a-2x)(a-2y)(a-2z) = 0 && \Rightarrow \\ &x = \frac{a}{2} \quad \vee \quad y = \frac{a}{2} \quad \vee \quad z = \frac{a}{2} \end{aligned}$$

This shows that P lies on one of medians of $\triangle ABC$. On the other hand, if P belongs to one of the medians, say on the A-median, then $z = \frac{a}{2}$ and by Ceva's theorem $xy = (a-x)(a-y)$, or $x + y = a$. Thus $x + y + z = \frac{3a}{2}$, so that the required condition holds.

Therefore the locus of the point P is the union of the interior points of the three medians of $\triangle ABC$. $\square$