Problema J174. The incircle of triangle ABC touches sides BC, CA, AB at D, E, F, respectively. Let K be a point on side BC and let M be the point on the line segment AK such that $AM = AE = AF$. Denote by L and N the incenters of triangles ABK and ACK, respectively. Prove that K is the foot of the altitude from A if and only if $DLMN$ is a square.

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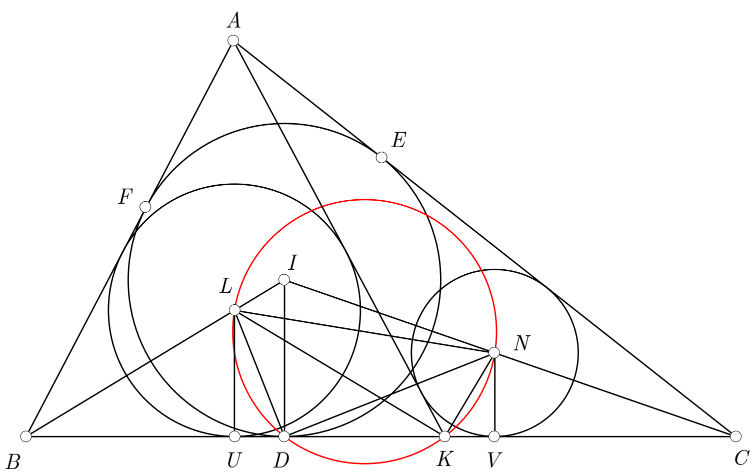
We begin by proving two lemmas.

Lemma 1. The points D, K lie on the circle with diameter LN.

Proof. Suppose wlog that $c < b$. Let I, U, V be the incenter of ABC and the points where the circles (L), (N) touch the side BC; let r, r_1, r_2 be the inradii of the circles (I), (L), (N), as shown in figure.



Denote $a = BC$, $b = CA$, $c = AB$, $m = BK$, $n = KC$, $x = AK$.
Since L and N are the incenters of $\triangle ABK$ and $\triangle ACK$ we have

$$\angle LKN = \angle LKA + \angle AKN = \frac{1}{2}(\angle BKA + \angle AKC) = 90^\circ$$

In order to prove that $\angle LDN = 90^\circ$ we will show that

$$LD^2 + DN^2 = LN^2 \tag{1}$$

The Pythagora's theorem yields $LD^2 = r_1^2 + UD^2$, $ND^2 = r_2^2 + DV^2$ and

$$LN^2 = UV^2 + (r_1 - r_2)^2 = UD^2 + DV^2 - 2UD \cdot DV + r_1^2 + r_2^2 - 2r_1r_2$$

Therefore to establish (3) it is enough to show that $UD \cdot DV = r_1r_2$.

We clearly have

$$UD = BD - BU = \frac{a+c-b}{2} - \frac{m+c-x}{2} = \frac{a+x-b-m}{2} \tag{2}$$

$$DV = DC - CV = \frac{a+b-c}{2} - \frac{m+b-x}{2} = \frac{a+x-c-n}{2} \tag{3}$$

From (2) and (3), plugging $n = a - m$ into the expression, we obtain

$$UD \cdot DV = \frac{(x+a-b-m)(x-c+m)}{4} \tag{4}$$

From the similar triangles $\triangle BUL \sim \triangle BDI$, $\triangle CVN \sim \triangle CDI$, it follows that

$$LU : ID = BU : BD \quad \Rightarrow \quad r_1 = r \cdot \frac{c+m-x}{a+c-b} \tag{5}$$

$$NV : ID = CV : CD \quad \Rightarrow \quad r_2 = r \cdot \frac{b+n-x}{a+b-c} \tag{6}$$

From (5),(6) using the well known formula

$$r^2 = \frac{(b+c-a)(a+c-b)(a+b-c)}{4(a+b+c)}$$

and plugging $n = a - m$ into the expression, we obtain

$$r_1 r_2 = \frac{(b+c-a)(a+b-m-x)(c+m-x)}{4(a+b+c)} \tag{7}$$

Using (4) and (7) we have

$$UD \cdot DV - r_1 r_2 = \frac{ax^2 - ac^2 + a^2 m - b^2 m + c^2 m - am^2}{2(a+b+c)} \tag{8}$$

From the Stewart's theorem we get

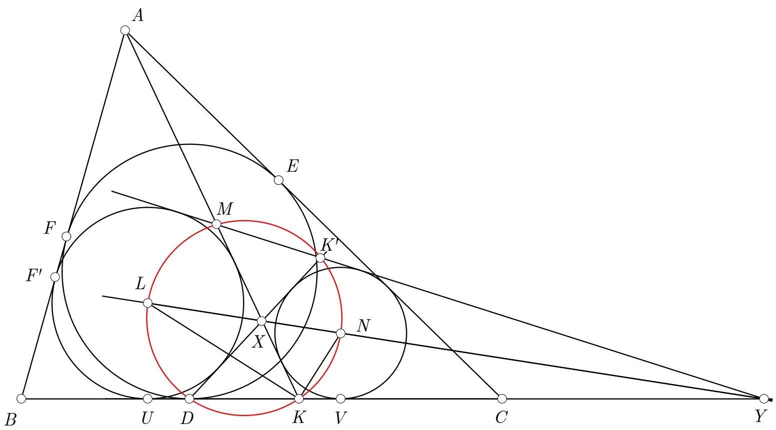
$$x^2 = \frac{mb^2 + (a-m)c^2 - am(a-m)}{a}$$

Finally, plugging x^2 into (8), after a boring calculation, we have

$$UD \cdot DV - r_1 r_2 = \frac{\left(c^2 - am\right)\left(m+n-a\right)}{2(a+b+c)} = 0$$

Thus $LD^2 + DN^2 = LN^2$ and the lemma is proved. ∎

Lemma 2. If M is the second intersection point of AK with the circle γ circumscribed to $DKNL$, then $DM \perp LN$ and $AM = AE = AF$.



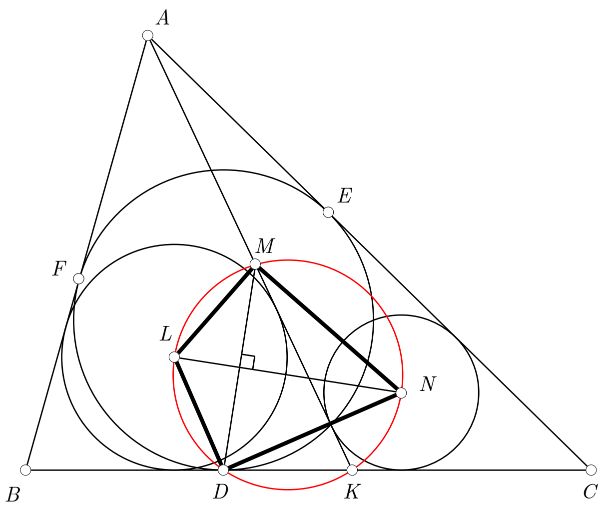
Proof. Let the incircle of triangle ABK touches side AB at F'. According to Lemma 1 the center of γ is the midpoint of LN, so the point M lies on the external tangent to the circles (L), (N). Therefore, because of simmetry of the figure, we have $DM \perp LN$ and

$$\begin{aligned} AM &= AF' - UD = AF' - (BD - BU) \\ &= \frac{c+x-m}{2} - \frac{a+c-b}{2} + \frac{c+m-x}{2} \\ &= \frac{b+c-a}{2} = AF \end{aligned}$$

The lemma is proved. ■

Considering now the original problem, from Lemma 1 and Lemma 2 it follows that

- $DLMN$ is cyclic;
- $\angle LDN = \angle LMN = 90°$;
- $DM \perp LN$.



Therefore $DLMN$ is a square if and only if MD is a diameter of the circumcircle of $DLMN$, i.e. $\angle MKD = 90°$ (i.e. $AK \perp BC$). □