Problema J217. If a, b, c are integers such that $a^2 + 2bc = 1$ and $b^2 + 2ca = 2012$, find all possible values of $c^2 + 2ab$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy

Subtracting the two given equalities from each other, we get

$$b^2 - a^2 + 2ac - 2bc = 2011 \quad \Leftrightarrow \quad (a - b)(2c - a - b) = 2011$$

Since 2011 is prime we have the following cases:

- $a - b = 1$, $2c - a - b = 2011$, $a^2 + 2bc = 1$ yielding

$$(a, b, c) \in \left\{ (1, 0, 1006), \left(-\frac{2001}{3}, -\frac{2014}{3}, \frac{1004}{3} \right) \right\}$$

- $a - b = -1$, $2c - a - b = -2011$, $a^2 + 2bc = 1$ yielding

$$(a, b, c) \in \left\{ (-1, 0, 1006), \left(\frac{2001}{3}, \frac{2014}{3}, -\frac{1004}{3} \right) \right\}$$

- $a - b = 2011$, $2c - a - b = 1$, $a^2 + 2bc = 1$ which has no solutions in $\mathbb{R}$.

- $a - b = -2011$, $2c - a - b = -1$, $a^2 + 2bc = 1$ which has no solutions in $\mathbb{R}$.

Since a, b, c are integers, the only acceptable values for a, b, c are

$$(a, b, c) \in \{(1, 0, 1006), (-1, 0, 1006)\}$$

so $c^2 + 2ab = 1006^2 = 1012036$. $\qquad\qquad\square$