Problema J218. Prove that in any triangle with sides of lenghts a, b, c, circumradius R, and inradius r, the following inequality holds

$$\frac{\sqrt{ab}}{a+b-c}+\frac{\sqrt{bc}}{b+c-a}+\frac{\sqrt{ca}}{c+a-b}\leq 1+\frac{R}{r}$$

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Let s be the semi-perimeter and let Δ be the area of triangle. According to the well known identities

$$S=\sqrt{s(s-a)(s-b)(s-c)},\quad r=\frac{S}{s},\quad R=\frac{abc}{4S}$$

we get

$$1+\frac{R}{r}=1+\frac{2abc}{(a+b-c)(b+c-a)(a+c-b)}$$

so the required inequality equivals to

$$\frac{\sqrt{ab}}{a+b-c}+\frac{\sqrt{bc}}{b+c-a}+\frac{\sqrt{ca}}{c+a-b}\leq 1+\frac{2abc}{(a+b-c)(b+c-a)(a+c-b)}$$

Using the Ravi transformation $a=y+z$, $b=x+z$, $c=x+y$ our inequality rewrites as

$$\frac{yz\sqrt{(x+y)(x+z)}+xz\sqrt{(x+y)(y+z)}+xy\sqrt{(x+z)(y+z)}}{2xyz}\leq 1+\frac{(x+y)(y+z)(z+x)}{4xyz}$$

Therefore it is enough to prove that

$$2yz\sqrt{(x+y)(x+z)}+2xz\sqrt{(x+y)(y+z)}+2xy\sqrt{(x+z)(y+z)}\leq 4xyz+(x+y)(y+z)(z+x)$$

which follows immediately from AM-GM inequality since

$$\begin{aligned}
&2yz\sqrt{(x+y)(x+z)}+2xz\sqrt{(x+y)(y+z)}+2xy\sqrt{(x+z)(y+z)}\\
\leq& yz(2x+y+z)+xz(x+2y+z)+xy(x+y+2z)\\
=&4xyz+(x+y)(y+z)(z+x)
\end{aligned}$$

This ends the proof. Equality holds for $a=b=c$. $\square$