

Problema J219. Trying to solve a problem, Jimmy used the following "formula": $\log_{ab} x = \log_a x \log_b x$, where a, b, x are positive real numbers different from 1. Prove that this is correct only if x is a solution to the equation $\log_a x + \log_b x = 1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy

Setting $u = \log_{ab} x$, $v = \log_a x$, $w = \log_b x$ we have

$$\log_{ab} x = \log_a x \log_b x \quad \Leftrightarrow$$

$$x = (ab)^u = a^u b^u = a^{vw} b^{vw} = (a^v)^w (b^w)^v = x^w x^v = x^{v+w} \quad \Leftrightarrow$$

$$v + w = 1 \quad \Leftrightarrow$$

$$\log_a x + \log_b x = 1$$

and the result is proved. □