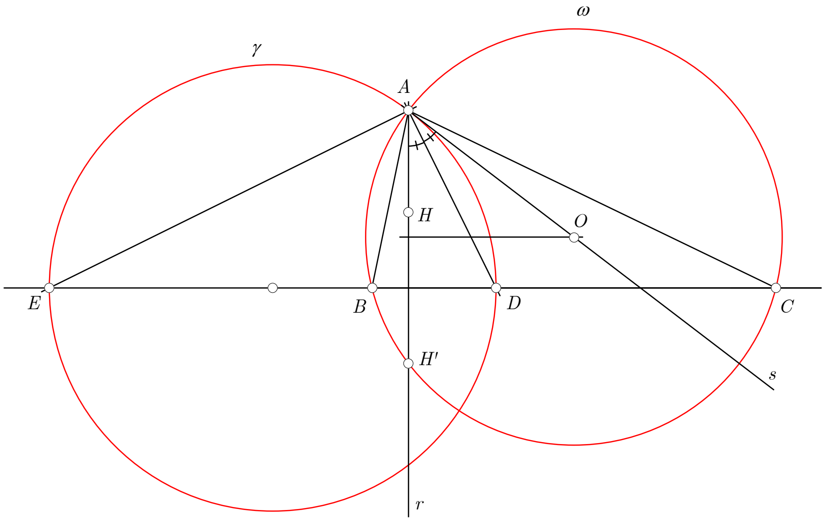
Problema J222. Give a ruler and straightedge construction of a triangle ABC given its orthocenter and the intersection points of the internal and external angle bisectors of one of its angles with the corresponding opposite side.

Proposed by Cosmin Pohoata, Princeton University, USA

Solution by Ercole Suppa, Teramo, Italy

ANALYSIS. Let ABC be the required triangle and denote by D, E the intersection points of the internal and external angle bisectors of $\angle BAC$ with BC. Let H be the orthocenter of ABC and draw a figure in order to find some relation between the given elements and the unknown parts.



Observe that the vertices B, C lie on the line DE, whereas the vertex A is the meeting point of the line through H perpendicular to DE with the circle having for diameter DE (since AE and AD are perpendicular).

Let O be the circumcentre of $\triangle ABC$ and note that $\angle BAH = 90° - B = \angle OAC$, so $\angle HAD = \angle DAO$. Therefore O belong to the line symmetric of AH with respect to AD.

Let H' be the symmetric of H with respect to DE. It is well known that H' lies on the circumcircle (O). Hence O belong to the mediator of segment AH'.

The vertices B, C are the intersection points of (O) with DE.

CONSTRUCTION. The vertices A, B, C of the required triangle can be constructed in the following way:

- draw the line DE;
- draw the circle γ having for diameter DE;
- draw the line r perpendicular to DE through H;
- construct the point $A = r \cap \gamma$;
- draw the line s symmetric of r wrt AD;
- construct the point H' symmetric of H wrt DE;

- draw the line a mediator of the segment AH';
- construct the point $O = a \cap s$;
- draw the circle ω through A and having center O;
- construct the points $\{B, C\} = \omega \cap DE$.

$\square$