Problema J236. Let ABC be a triangle and let $ABRS$ and $ACXY$ be the two squares constructed on sides AB and AC which are directed towards the exterior of the triangle. If U is the circumcenter of triangle SAY, prove that the line AU is the A-symmedian of triangle ABC.

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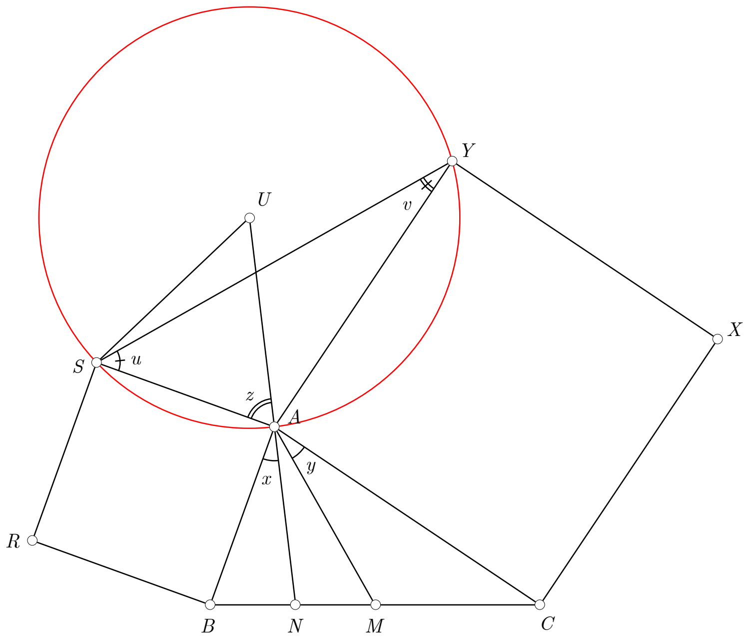

Let $a = BC$, $b = CA$, $c = AB$, let A, B, C be the three angles of $\triangle ABC$, let $N = AU \cap BC$ and denote by M the midpoint of BC. Let $x = \angle BAN$, $y = \angle MAC$, $z = \angle SAU$, $u = \angle ASY$ and $v = \angle AYS$ as shown in figure.

By using the Sine Law we get

$$\begin{aligned}
&1 = \frac{BM}{MC} = \frac{AB}{AC} \cdot \frac{\sin \angle BAM}{\sin \angle MAC} && \Leftrightarrow \\
&c \sin(A - y) = b \sin y && \Leftrightarrow \\
&c \sin A \cos y - c \cos A \sin y = b \sin y && \Leftrightarrow \\
&c \sin A \cos y - c \cos A \tan y = b \tan y
\end{aligned}$$

hence

$$\tan y = \frac{c \sin A}{b + c \cos A} \tag{1}$$

Applying the Sine Law in triangle $\triangle SAY$ yields

$$\begin{aligned}
&\frac{AS}{\sin v} = \frac{AY}{\sin u} && \Leftrightarrow \\
&c \sin u = b \sin v && \Leftrightarrow \\
&c \sin(A - v) = b \sin v && \Leftrightarrow \\
&c \sin A \cos v - c \cos A \sin v = b \sin v && \Leftrightarrow \\
&c \sin A \cos y - c \cos A \tan v = b \tan v &&
\end{aligned}$$

hence

$$\tan v = \frac{c \sin A}{b + c \cos A} \tag{2}$$

From (1) and (2) it follows that $v = y$. On the other hand clearly we have

$$x = 90^\circ - z = 90^\circ - (90^\circ - v) = v$$

Therefore $x = y$ and this means that AU is the A-symmedian of triangle ABC, as required. $\square$