Problema J237. Prove that the diameter of the incircle of a triangle ABC is equal to $\frac{AB-BC+CA}{\sqrt{3}}$ if and only if $\angle BAC = 60°$.

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It is well known that $r = (s-a)\tan\frac{A}{2}$ where as usual r, s denote the inradius and the semiperimeter of $\triangle ABC$ with sidelengths a, b, c. Therefore the diameter of the incircle of $\triangle ABC$ is equal to $\frac{AB-BC+CA}{\sqrt{3}}$ if and only if

$$2(s-a)\tan\frac{A}{2} = \frac{b+c-a}{\sqrt{3}} \qquad\qquad \Leftrightarrow$$

$$(b+c-a)\tan\frac{A}{2} = \frac{b+c-a}{\sqrt{3}} \qquad\qquad \Leftrightarrow$$

$$\tan\frac{A}{2} = \frac{1}{\sqrt{3}} \qquad\qquad \Leftrightarrow$$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} = \frac{\frac{2}{\sqrt{3}}}{1-\frac{1}{3}} = \sqrt{3} \qquad\qquad \Leftrightarrow$$

$$\angle BAC = 60°$$

$\square$

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