Problem 338. Sequences $\{a_n\}$ and $\{b_n\}$ satisfies $a_0 = 1$, $b_0 = 0$ and for $n = 0, 1, 2, \ldots$

$$a_{n+1} = 7a_n + 6b_n - 3$$
$$b_{n+1} = 8a_n + 7b_n - 4$$

Prove that a_n is a perfect square for all $n = 0, 1, 2, \ldots$.

Solution by Ercole Suppa, Teramo, Italy. By getting b_n from the first equation and replacing it into the second one, we have

$$a_{n+2} = 14a_{n+1} - a_n - 6, \qquad n = 0, 1, 2, \ldots \tag{*}$$

with $a_0 = 1$, $a_1 = 4$. Now, consider the sequence $\{c_n\}$ of positive integers, defined by $c_0 = 1$, $c_1 = 2$ and

$$c_{n+2} = 4c_{n+1} - c_n, \qquad n = 0, 1, 2, \ldots \tag{**}$$

Solving (*) and (**) we get

$$a_n = \frac{1}{4}\left[2 + \left(7 - 4\sqrt{3}\right)^n + \left(7 + 4\sqrt{3}\right)^n\right]$$
$$c_n = \frac{1}{2}\left[\left(2 - \sqrt{3}\right)^n + \left(2 + \sqrt{3}\right)^n\right]$$

from which easily follows that $a_n = c_n^2$ for all $n = 0, 1, 2, \ldots$, proving the desired result.