Problem 383. Let O and I be the circumcenter and incenter of $\triangle ABC$ respectively. If $AB \neq AC$ and $BC = (AB + AC)/2$ then prove that the line OI and the bisector of $\angle CAB$ are perpendicular.

Solution by Ercole Suppa, Teramo, Italy.

Let $a = BC$, $b = CA$, $c = AB$ and let R, r, s, be the circumradius, the inradius and the semiperimeter of $\triangle ABC$ respectively. Using the well known formulas $OI^2 = R^2 - 2Rr$, $s - a = AI \cdot \cos \frac{A}{2}$, $Rr = \frac{abc}{4s}$ and $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ we have

$$AI^2 = \frac{(s-a)^2}{\frac{s(s-a)}{bc}} = \frac{bc(s-a)}{s} = \frac{bc(b+c-a)}{a+b+c}$$

$$OI^2 = R^2 - 2Rr = R^2 - \frac{abc}{a+b+c}$$

If $b + c = 2a$, we get $AI^2 = \frac{abc}{a+b+c}$ so

$$AI^2 + OI^2 = R^2 = OA^2 \quad \Rightarrow \quad AI \perp OI$$

and we are done. $\square$