An inequality for the excircles of a triangle **October 2009**

1829. *Proposed by Oleh Faynshteyn, Leipzig, Germany.*

Let ABC be a triangle with $BC = a$, $CA = b$, and $AB = c$. Let r_a denote the radius of the excircle tangent to BC, r_b the radius of the excircle tangent to CA, and r_c the radius of the excircle tangent to AB. Prove that

$$\frac{r_a r_b}{(a+b)^2} + \frac{r_b r_c}{(b+c)^2} + \frac{r_c r_a}{(c+a)^2} \leq \frac{9}{16}.$$

I. *Solution by Ercole Suppa, Teramo, Italy.*

Let $\Delta = \text{Area}(ABC)$ and $s = (a+b+c)/2$. Taking into account the identities $r_a = \Delta/(s-a)$, $r_b = \Delta/(s-b)$, $r_c = \Delta/(s-c)$, as well as Heron's formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, the inequality in the problem becomes

$$\begin{aligned} \frac{s(s-c)}{(a+b)^2} + \frac{s(s-a)}{(b+c)^2} + \frac{s(s-c)}{(c+a)^2} &\leq \frac{9}{16} && \Leftrightarrow \\ \frac{(a+b)^2 - c^2}{(a+b)^2} + \frac{(b+c)^2 - a^2}{(b+c)^2} + \frac{(c+a)^2 - b^2}{(c+a)^2} &\leq \frac{9}{4} && \Leftrightarrow \\ \frac{a^2}{(b+c)^2} + \frac{b^2}{(a+c)^2} + \frac{c^2}{(a+b)^2} &\geq \frac{3}{4}. && (1) \end{aligned}$$

Now, by using Nesbitt's inequality

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

and the well known inequality $3\left(x^2+y^2+z^2\right) \geq (x+y+z)^2$ we get

$$\begin{aligned} \left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2 &\geq \frac{1}{3}\left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}\right)^2 \geq \\ &\geq \frac{1}{3}\left(\frac{3}{2}\right)^2 = \frac{3}{4} \end{aligned}$$

and (1) is proved. The equality holds if and only if $a = b = c$.