Problema A2. Suppose that $ABCD$ is a quadrilateral which circumscribes the circle (O). The tangential points of (O) with AB, BC, CD, DA are M, N, P, Q respectively. The perpendicular line with MN through M cuts PQ at I. The perpendicular line with MQ through M cuts PN at J. Prove that $AI \| BJ$.

Proposed by Hoang Quoc Khanh, Vinh Phuc highschool for Gifted students, Vinh Phuc province.

Solution by Ercole Suppa, Teramo, Italy

First we prove the following lemma:

Lemma. Let ABC be a triangle inscribed in the circle (O). Let X be an arbitrary point on AB and $Y = OX \cap AC$. Let t be the tangent at A to the circumcircle. The lines through O parallel to AB and AC intersect t at U, V respectively. Prove that $UX \| VY$.

Proof. Let $D = AB \cap OU$, $E = AC \cap OV$ and $F = AB \cap VY$.

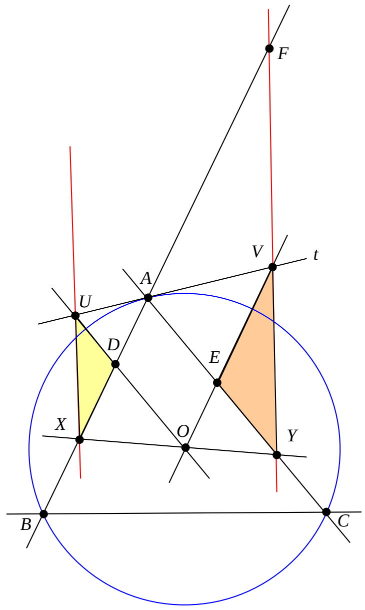

Figure 1

Taking into account the equality of the opposite sides of the parallelogram $ADOE$ and the similarity $\triangle UDA \sim \triangle AEV$, $\triangle DXO \sim \triangle EOY$, we have:

$$\frac{UD}{DX} = \frac{UD}{AD} \cdot \frac{AD}{DX} = \frac{AE}{EV} \cdot \frac{EO}{DX} = \frac{AE}{EV} \cdot \frac{EY}{DO} = \frac{AE}{EV} \cdot \frac{EY}{AE} = \frac{EY}{EV}$$

Therefore the triangles $\triangle DUX$ and $\triangle EYV$ are similar, hence

$$\angle UXD = \angle EVY = \angle AFY$$

and the proof of the lemma is complete. ■

Coming back to the problem denote $R = MI \cap (O)$ and $S = MJ \cap (O)$ as shown in FIGURE 2.

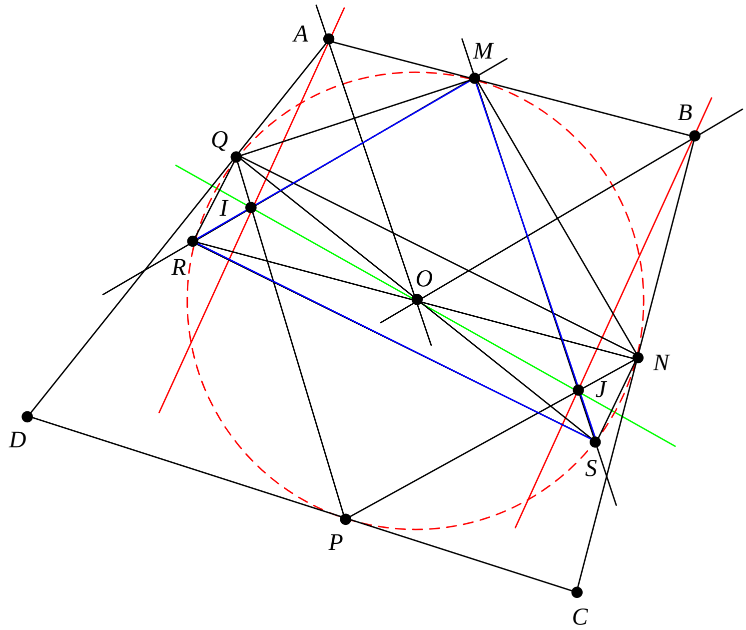

FIGURE 2

We can notice that $QS \cap RN = O$ because $\angle QMS = \angle RMN = 90°$, so the Pascal theorem applied to the hexagon $QPNRMS$ yields that I, O, J are collinear. Thus the result follows at once from the LEMMA applied to the triangle $\triangle MRS$ since $AO \| MS$, $BO \| MR$. $\square$