Problema A3. Given ABC is a triangle with ℓ_a, ℓ_b, ℓ_c are the bisectors of A, B, C respectively. Provethat the following inequality holds

$$\frac{a+b}{\ell_a+\ell_b}+\frac{b+c}{\ell_b+\ell_c}+\frac{c+a}{\ell_c+\ell_a}\geq 2\sqrt{3}$$

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Let $s=\frac{a+b+c}{2}$. From the well known formula of the angle bisector we have

$$\ell_a=\frac{2bc}{b+c}\cos\frac{A}{2}=\frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}}\leq\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}\leq\sqrt{s(s-a)}$$

and similar inequalities hold for ℓ_b, ℓ_c. Thus from AM-QM inequality we get

$$\begin{aligned}\ell_a+\ell_b\leq\sqrt{s(s-a)}+\sqrt{s(s-b)}\leq 2\sqrt{\frac{s(s-a)+s(s-b)}{2}}\leq\\ \leq\sqrt{2s(2s-a-b)}=\sqrt{(a+b+c)c}\end{aligned}$$

Therefore, in order to demonstrate the proposed inequality, is enough to prove that

$$\frac{a+b}{\sqrt{(a+b+c)c}}+\frac{b+c}{\sqrt{(a+b+c)a}}+\frac{c+a}{\sqrt{(a+b+c)b}}\geq 2\sqrt{3} \qquad (*)$$

WLOG we can assume that $a+b+c=1$. Then the inequality (*) becomes

$$\frac{1-a}{\sqrt{a}}+\frac{1-b}{\sqrt{b}}+\frac{1-c}{\sqrt{c}}\geq 2\sqrt{3} \qquad (**)$$

The inequality (**) follows directly from Jensen's inequality, because the function:

$$f(x)=\frac{1-x}{\sqrt{x}}$$

is convex on $[0,1]$ since:

$$f''(x)=\frac{x+3}{4x\sqrt{x}}\geq 0 \qquad , \qquad \forall x\in(0,1]$$

So the proof is complete. □

Remark. The inequality (**) can be proved also by means of classic inequalities, in the following way. By Cauchy-Schwartz inequality we have

$$\sqrt{a}+\sqrt{b}+\sqrt{b}\leq\sqrt{1+1+1}\sqrt{a+b+c}=\sqrt{3}$$

Thus, taking into account the inequality between harmonic and arithmetic means, we get:

$$\begin{aligned}\frac{1-a}{\sqrt{a}}+\frac{1-b}{\sqrt{b}}+\frac{1-c}{\sqrt{c}}&=\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}-\left(\sqrt{a}+\sqrt{b}+\sqrt{b}\right)\geq\\ &\geq\frac{9}{\sqrt{a}+\sqrt{b}+\sqrt{b}}-\left(\sqrt{a}+\sqrt{b}+\sqrt{b}\right)\geq\\ &\geq\frac{9}{\sqrt{3}}-\sqrt{3}=2\sqrt{3}\end{aligned}$$

and (**) is proven.