

MathContest - Problem 3. Let m_a, m_b, m_c and R be the medians and the circum-radius of a triangle ABC . Prove that

$$\frac{m_a^2 + m_b^2 + m_c^2}{R^2 (\sin^2 A + \sin^2 B + \sin^2 C)}$$

is a positive integer and determine its value.

Solution by Ercole Suppa, Teramo, Italy

By using the Apollonius's formula

$$m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$$

and its analogs we have

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4} (a^2 + b^2 + c^2) \quad (1)$$

On the other hand the extended sine law yields $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$. Therefore

$$R^2 (\sin^2 A + \sin^2 B + \sin^2 C) = \frac{1}{4} (a^2 + b^2 + c^2) \quad (2)$$

From (1) and (2) it follows

$$\frac{m_a^2 + m_b^2 + m_c^2}{R^2 (\sin^2 A + \sin^2 B + \sin^2 C)} = 3$$

and the problem is solved. □