

Problema O111. Prove that for each positive integer n the number

$$\left(\binom{n}{0} + 2\binom{n}{2} + 2^2\binom{n}{4} + \cdots \right)^2 \cdot \left(\binom{n}{1} + 2\binom{n}{3} + 2^2\binom{n}{5} + \cdots \right)^2$$

is triangular.

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We denote

$$f(n) = \left(\binom{n}{0} + 2\binom{n}{2} + 2^2\binom{n}{4} + \cdots \right)^2 \cdot \left(\binom{n}{1} + 2\binom{n}{3} + 2^2\binom{n}{5} + \cdots \right)^2$$

For each real number x the the Binomial Theorem yields:

$$(x+1)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \quad (1)$$

$$(x-1)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \cdots \quad (2)$$

By summing (1) and (2) we have

$$\frac{1}{2} [(x+1)^n + (x-1)^n] = \binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \cdots \quad (3)$$

$$\frac{1}{2x} [(x+1)^n - (x-1)^n] = \binom{n}{1} + \binom{n}{3}x^2 + \binom{n}{5}x^4 \cdots \quad (4)$$

From (3) and (4), taking $x = \sqrt{2}$, we obtain:

$$\frac{1}{2} [(\sqrt{2}+1)^n + (\sqrt{2}-1)^n] = \binom{n}{0} + \binom{n}{2}2 + \binom{n}{4}2^2 + \cdots \quad (5)$$

$$\frac{1}{2\sqrt{2}} [(\sqrt{2}+1)^n - (\sqrt{2}-1)^n] = \binom{n}{1} + \binom{n}{3}2 + \binom{n}{5}2^2 + \cdots \quad (6)$$

Thus

$$f(n) = \left(\frac{a^n + b^n}{2} \right)^2 \cdot \left(\frac{a^n - b^n}{2\sqrt{2}} \right)^2 = \frac{1}{2} \left(\frac{a^n + b^n}{2} \right)^2 \cdot \left(\frac{a^n - b^n}{2} \right)^2 \quad (7)$$

where $a = \sqrt{2} + 1$ and $b = \sqrt{2} - 1$. Let $k = \left(\frac{a^n - b^n}{2} \right)^2$. Since $ab = 1$ we have:

$$\begin{aligned} k+1 &= \left(\frac{a^n - b^n}{2} \right)^2 + 1 = \frac{a^{2n} + b^{2n} - 2a^n b^n}{4} + 1 = \\ &= \frac{a^{2n} + b^{2n} - 2 + 4}{4} = \frac{a^{2n} + b^{2n} + 2a^n b^n}{4} = \left(\frac{a^n + b^n}{2} \right)^2 \end{aligned}$$

Therefore

$$f(n) = \frac{k(k+1)}{2} = \binom{k+1}{2}$$

and we are done. \square