

Problema O112. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + abc}{(b+c)^2} + \frac{b^3 + abc}{(c+a)^2} + \frac{c^3 + abc}{(a+b)^2} \geq \frac{3}{2} \cdot \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2}$$

Proposed by Cezar Lupu, University of Bucharest, Romania and Pham Huu Duc, Ballajura, Australia

Solution by Ercole Suppa, Teramo, Italy

The inequality can be rewritten in the following form:

$$\begin{aligned} \sum_{cyc} \left[\frac{a^3 + abc}{(b+c)^2} - \frac{a}{2} \right] &\geq \frac{3}{2} \cdot \frac{a^3 + b^3 + c^3}{a^2 + b^2 + c^2} - \frac{a+b+c}{2} & \Leftrightarrow \\ \sum_{cyc} \frac{a(2a^2 - b^2 - c^2)}{2(b+c)^2} &\geq \sum_{cyc} \frac{a^3 - a^2b}{2(a^2 + b^2 + c^2)} + \sum_{cyc} \frac{a^3 - ab^2}{2(a^2 + b^2 + c^2)} & \Leftrightarrow \\ \sum_{cyc} \frac{a(a^2 - b^2)}{(b+c)^2} + \sum_{cyc} \frac{a(a^2 - c^2)}{(b+c)^2} &\geq \sum_{cyc} \left[\frac{a^2(a-b) + b^2(b-a)}{a^2 + b^2 + c^2} \right] & \Leftrightarrow \\ \sum_{cyc} \frac{a(a^2 - b^2)}{(b+c)^2} + \sum_{cyc} \frac{b(b^2 - a^2)}{(c+a)^2} &\geq \sum_{cyc} \frac{(a^2 - b^2)(a-b)}{a^2 + b^2 + c^2} & \Leftrightarrow \\ \sum_{cyc} \left[\frac{a}{(b+c)^2} - \frac{b}{(a+c)^2} \right] (a^2 - b^2) &\geq \sum_{cyc} \frac{a+b}{a^2 + b^2 + c^2} (a-b)^2 & \Leftrightarrow \\ \sum_{cyc} \frac{(a+b)(a^2 + b^2 + c^2 + ab + 2bc + 2ac)}{(a+c)^2(b+c)^2} (a-b)^2 &\geq \sum_{cyc} \frac{a+b}{a^2 + b^2 + c^2} (a-b)^2 & \Leftrightarrow \\ \sum_{cyc} \left[\frac{(a+b)(a^2 + b^2 + c^2 + ab + 2bc + 2ac)}{(a+c)^2(b+c)^2} - \frac{a+b}{a^2 + b^2 + c^2} \right] (a-b)^2 &\geq 0 & \Leftrightarrow \\ \sum_{cyc} \left[\frac{a^2 + b^2 + c^2 + ab + 2bc + 2ac}{(a+c)^2(b+c)^2} - \frac{1}{a^2 + b^2 + c^2} \right] (a-b)^2 &\geq 0 \end{aligned}$$

Denote

$$S_c = \frac{a^2 + b^2 + c^2 + ab + 2bc + 2ac}{(a+c)^2(b+c)^2} - \frac{1}{a^2 + b^2 + c^2}$$

and S_a, S_b are determined similarly. We need to prove that:

$$S_a(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2 \geq 0 \quad (*)$$

In order to apply the SOS method¹ for cyclic inequalities, we must consider the following cases:

¹See Pham Kim Hung, *Secrets in inequalities*, volume 2, GIL Publishing House

(i) If $a \geq b \geq c$ we have

$$S_b = \frac{a^4 + 2a^3b + a^2b^2 + a^3c - 3ab^2c + a^2c^2 + b^2c^2 + ac^3 + 2bc^3 + c^4}{(a+b)^2(b+c)^2(a^2+b^2+c^2)} \geq 0$$

since $2a^3b + a^2b^2 - 3ab^2c \geq 0$. Furthermore

$$S_a + S_b = \frac{f(a, b, c)}{(a+b)^2(a+c)^2(b+c)^2(a^2+b^2+c^2)} \geq 0$$

$$S_b + S_c = \frac{g(a, b, c)}{(a+b)^2(a+c)^2(b+c)^2(a^2+b^2+c^2)} \geq 0$$

since

$$\begin{aligned} f(a, b, c) = & a^6 + 2a^5b + a^4b^2 + a^2b^4 + 2ab^5 + b^6 + 3a^5c + 4a^4bc - a^3b^2c - a^2b^3c + 4ab^4c \\ & + 3b^5c + 4a^4c^2 + 2a^3bc^2 - 8a^2b^2c^2 + 2ab^3c^2 + 4b^4c^2 + 4a^3c^3 + a^2bc^3 \\ & + ab^2c^3 + 4b^3c^3 + 5a^2c^4 + 8abc^4 + 5b^2c^4 + 5ac^5 + 5bc^5 + 2c^6 \end{aligned}$$

with

$$2a^5b - a^3b^2c - a^2b^3c \geq 0 \quad , \quad 4a^4bc + 4a^4c^2 - 8a^2b^2c^2 \geq 0$$

and

$$\begin{aligned} g(a, b, c) = & 2a^6 + 5a^5b + 5a^4b^2 + 4a^3b^3 + 4a^2b^4 + 3ab^5 + b^6 + 5a^5c + 8a^4bc + a^3b^2c \\ & + 2a^2b^3c + 4ab^4c + 2b^5c + 5a^4c^2 + a^3bc^2 - 8a^2b^2c^2 - ab^3c^2 + b^4c^2 \\ & + 4a^3c^3 + 2a^2bc^3 - ab^2c^3 + 4a^2c^4 + 4abc^4 + b^2c^4 + 3ac^5 + 2bc^5 + c^6 \end{aligned}$$

with

$$5a^5b + 5a^4b^2 - 8a^2b^2c^2 - ab^3c^2 - ab^2c^3 \geq 0$$

(ii) If $c \geq b \geq a$ in a similar way we may prove that

$$S_b \geq 0 \quad , \quad S_a + S_b \geq 0 \quad , \quad S_b + S_c \geq 0$$

According to one of criteria of SOS method the inequality (*) is proved. Equality holds for $a = b = c$. \square