Problema O115. Numbers 1 through 24 are written on a board. At any time, numbers a, b, c may be replaced by

$$\frac{2b+2c-a}{3} \quad , \quad \frac{2c+2a-b}{3} \quad , \quad \frac{2a+2b-c}{3}$$

Can a number greater than 70 eventually appear on the board ?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy

Let $S_n = \left\{x_1^{(n)}, \ldots, x_{24}^{(n)}\right\}$ be the numbers written on the board after n iterations. For any three real numbers a, b, c we have

$$\left(\frac{2b+2c-a}{3}\right)^2 + \left(\frac{2c+2a-b}{3}\right)^2 + \left(\frac{2a+2b-c}{3}\right)^2 = a^2+b^2+c^2$$

Therefore the function

$$I(n) = \max\left\{a^2+b^2+c^2 \mid a,b,c \in S_n\right\}$$

is an invariant, i.e. $I(n)$ does not change during the whole process. Since

$$I(1) = 22^2+23^2+24^2 = 1589$$

a number greater than 70 can not appear on the board (after n steps) because we would have

$$I(n) \geq 70^2 = 4900 > 1589$$

This is a contradiction and the result follows. $\square$