Problema O118. Solve in positive integers the equation

$$x^2 + y^2 + z^2 - xy - yz - zx = w^2$$

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We use the following

LEMMA. All integral solutions of the diophantine equation

$$x^2 + xy + y^2 = z^2$$

are given by

$$\begin{cases} x = k\left(n^2 + 2mn\right) \\ y = k\left(m^2 - n^2\right) \\ z = k\left(m^2 + n^2 + mn\right) \end{cases} , \qquad \begin{cases} x = k\left(m^2 - n^2\right) \\ y = k\left(n^2 + 2mn\right) \\ z = k\left(m^2 + n^2 + mn\right) \end{cases}$$

where $k, m, n \in \mathbb{Z}$.

Proof. The proof can be found in the book: *Titu Andreescu and Dorin Andrica, An introduction to diophantine equations.* ■

Coming back to the problem, the given equation can be written in the form:

$$(x - y)^2 + (y - z)^2 + (z - x)^2 = 2w^2 \tag{1}$$

By setting $X = x - z$, $Y = y - z$ the equation (1) rewrites as

$$X^2 + Y^2 + (X + Y)^2 = 2w^2$$

$$X^2 + XY + Y^2 = w^2$$

By the LEMMA there are $k, m, n \in \mathbb{Z}$ such that

$$\begin{cases} X = k\left(n^2 + 2mn\right) \\ Y = k\left(m^2 - n^2\right) \\ w = k\left(m^2 + n^2 + mn\right) \end{cases} , \qquad \begin{cases} X = k\left(m^2 - n^2\right) \\ Y = k\left(n^2 + 2mn\right) \\ w = k\left(m^2 + n^2 + mn\right) \end{cases}$$

By solving the system

$$\begin{cases} x - y = X \\ y - z = Y \\ z - x = -X - Y \end{cases}$$

we get $x = X + Y + h$, $y = Y + h$, $z = h$, where $h \in \mathbb{Z}$. Thus the positive integers solutions of the our equation, up to permutations of x, y, z, are given by:

$$\begin{cases} x = k\left(m^2 + 2mn\right) + h \\ y = k\left(m^2 - n^2\right) + h \\ z = h \\ w = k\left(m^2 + n^2 + mn\right) \end{cases} \quad , \quad \begin{cases} x = k\left(m^2 - n^2\right) + h \\ y = k\left(n^2 + 2mn\right) + h \\ z = h \\ w = k\left(m^2 + n^2 + mn\right) \end{cases}$$

where $h, k, m, n \in \mathbb{N}_0$, $m > n$. $\square$