Problema O123. Let ABC be a triangle and let A_1, B_1, C_1 be the points of tangency of its incircle ω with triangle's sides. Medians A_1M, B_1N, C_1P in triangle $A_1B_1C_1$ intersect ω at A_2, B_2, C_2 respectively. Prove that AA_2, BB_2, CC_2 are concurrent at the isogonal conjugate of the Gergonne point Γ.

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We begin by proving the following

LEMMA. The tangents at the extremities of a chord PQ of a circle ω meet at A; any line through A cuts the circle at B, C, where A and C lie on opposite sides of line PQ; the line through C and the mid-point M of PQ intersects ω again in D. Then the lines AB, AD are isogonal conjugates.

Proof.
Let O be the center of ω and $R = AB \cap PQ$.

Claim 1. PQ bisects angle $\angle BMC$. (FIG. 1)
Proof. Since $AO \perp PQ$ it is enough to prove that $\angle BMA = \angle CMO$.

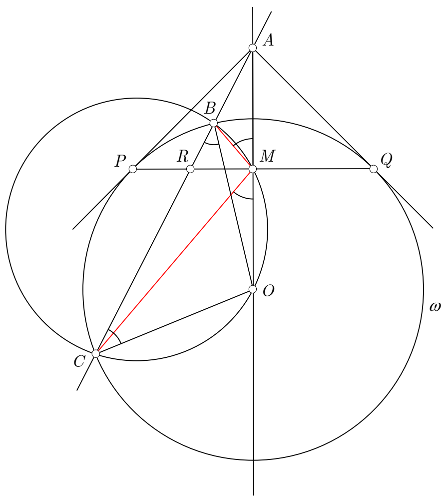

FIG. 1

By Euclid and the Power of a Point theorems we have

$$AO \cdot AM = AP^2 = AB \cdot AC$$

Thus the quadrilateral $BMOC$ is cyclic and, since $OB = OC$, we get

$$\angle BMA = 180^\circ - \angle BMO = \angle BCO = \angle CBO = \angle CMO$$

establishing the claim. ■

Claim 2. The points B, M, E are collinear. (FIG. 2)
Proof.

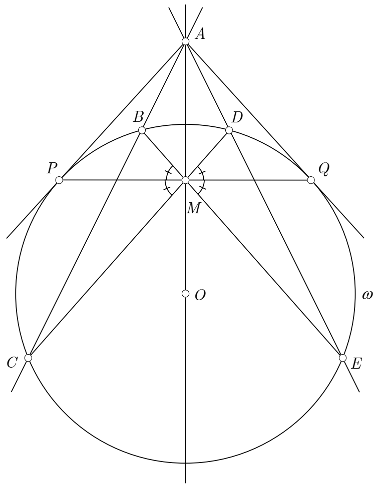

Fig. 2

By *Claim 1* we have

$$\angle BMP = \angle PMC \qquad , \qquad \angle DMQ = \angle QME$$

Since $\angle PMC = \angle DMQ$ we get $\angle BMP = \angle QME$, which proves our claim. ■

Claim 3. $\angle CAM = \angle EAM$. (Fig. 3)
Proof. Let us consider triangles $\triangle ABM$ and $\triangle ADM$.

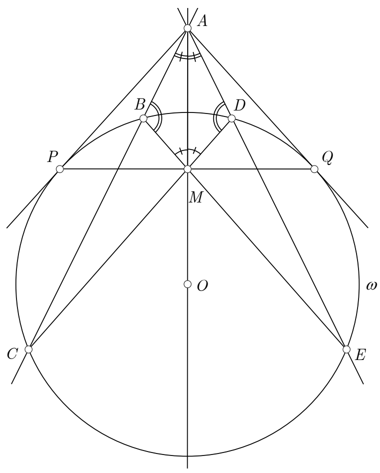

Fig. 3

The *Claim 1* yields

$$\angle PMB = \angle PMC = \angle DMQ \qquad \Rightarrow \qquad \angle AMB = \angle AMD$$

and, since B, C, E, D are concyclic, we have

$$\angle CBE = \angle CDE \qquad \Rightarrow \qquad \angle ABM = \angle ADM$$

Then we get $\angle CAM = \angle EAM$ and the the Lemma is proven. ■

Now by using the LEMMA we can easily solve the given problem. (FIG. 4)

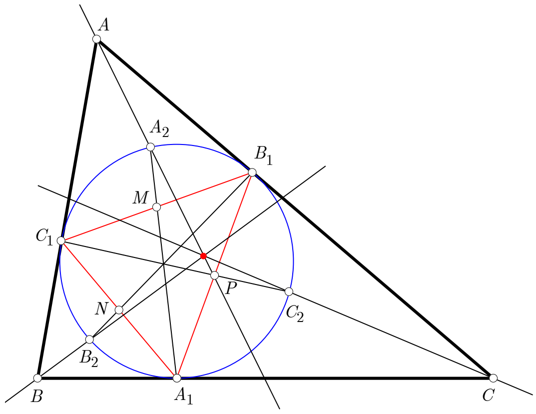

FIG. 4

In fact the pairs of lines AA_1 and AA_2, BB_1 and BB_2, CC_1 and CC_2, are isogonal conjugate and since AA_1, BB_1, CC_1 are concurrent at the Gergonne point Γ, the lines AA_2, BB_2, CC_2 are concurrent too at the point Γ' isogonal conjugate of Γ (due to a well known result). This ends the proof. $\square$