Problema O124. Let $S(n)$ be the number of pairs of positive integers (x, y) such that $xy = n$ and $\gcd(x, y) = 1$. Prove that

$$\sum_{d|n} S(d) = \tau\left(n^2\right)$$

where $\tau(s)$ is the number of divisors of s

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For notational convenience denote, for every $n \in \mathbb{N}$:

$$\Omega(n) = \left\{(x, y) \in \mathbb{N}^2 \mid xy = n,\ \gcd(x, y) = 1\right\} \qquad , \qquad f(n) = \sum_{d|n} S(d)$$

in such a way that $S(n) = |\Omega(n)|$.

Claim 1: If p is a prime number and $\alpha \in \mathbb{N}$ then $S(p^\alpha) = 2$.

Proof. From $\Omega\left(p^\alpha\right) = \left\{\left(p^\alpha, 1\right), \left(1, p^\alpha\right)\right\}$ follows $S(p^\alpha) = |\Omega(p^\alpha)| = 2$. ■

Claim 2: If p is a prime number and α, m are relatively prime positive integers, then $S(p^\alpha \cdot m) = 2 \cdot S(m)$.

Proof. Let us denote

$$A = \{(p^\alpha x, y) \mid (x, y) \in \Omega(n)\} \qquad , \qquad B = \{(x, p^\alpha y) \mid (x, y) \in \Omega(n)\}$$

Clearly $|A| = |B| = |\Omega(m)|$, $A \cap B = \emptyset$ and $A \cup B = \Omega\left(p^\alpha \cdot m\right)$, so by sum rule

$$S\left(p^\alpha \cdot m\right) = \left|\Omega\left(p^\alpha \cdot m\right)\right| = |A| + |B| = 2 \cdot |\Omega(m)| = 2 \cdot S(m) \qquad ■$$

Claim 3: If p is a prime number and α, m are positive integers with $\gcd(p, m) = 1$, then $f(p^\alpha \cdot m) = (2\alpha + 1) f(m)$.

Proof. If $\{d_1, d_2, \ldots, d_k\}$ are the divisors of m, by using the *Claim* 2, we have:

$$\begin{aligned} f(p^\alpha \cdot m) &= \sum_{d|p^\alpha \cdot m} S(d) = \sum_{i=1}^{k} S(d_i) + \sum_{i=1}^{k} S(p \cdot d_i) + \cdots + \sum_{i=1}^{k} S(p^\alpha \cdot d_i) = \\ &= f(m) + 2 \cdot f(m) + \cdots + 2 \cdot f(m) = \left(1 + \underbrace{2 + \cdots + 2}_{\alpha \text{ times}}\right) f(m) = \\ &= (2\alpha + 1) f(m) \end{aligned} \qquad ■$$

Consider now any positive integer n and suppose that its prime factorization is

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

Then the *Claim* 3 yields

$$\begin{aligned}\sum_{d|n} S(d) = f(n) = f\left(p_1^{\alpha_1} p_2^{\alpha_2} \cdot p_k^{\alpha_k}\right) = \\ &= (2\alpha_1 + 1)\, f\left(p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}\right) = \\ &\vdots \\ &= (2\alpha_1 + 1)\,(2\alpha_2 + 1) \cdots (2\alpha_k + 1)\end{aligned}$$

On the other hand, since $n^2 = p_1^{2\alpha_1} p_2^{2\alpha_2} \cdots p_k^{2\alpha_k}$, the number of divisors of n^2 is

$$\tau(n^2) = (2\alpha_1 + 1)\,(2\alpha_2 + 1) \cdots (2\alpha_k + 1)$$

establishing the desired result. $\square$