Problema O125. Let a, b, c be positive real numbers. Prove that

$$4 \leq \frac{a+b+c}{\sqrt[3]{abc}} + \frac{8abc}{(a+b)(b+c)(c+a)}$$

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Without loss of generality we may assume $abc = 1$, since the right expression is homogeneous. The inequality becomes

$$a+b+c+\frac{8}{(a+b)(b+c)(c+a)} - 4 \geq 0$$

By **AM-GM** inequality we have

$$a+b+c = \frac{(a+b)+(b+c)+(c+a)}{2} \geq \frac{3}{2}\sqrt[3]{(a+b)(b+c)(c+a)} \qquad (1)$$

After setting $(a+b)(b+c)(c+a) = x^3$, the inequality (1) yields

$$a+b+c+\frac{8}{(a+b)(b+c)(c+a)} - 4 \geq$$

$$\geq \frac{3}{2}\sqrt[3]{(a+b)(b+c)(c+a)} + \frac{8}{(a+b)(b+c)(c+a)} - 4 =$$

$$= \frac{3}{2}x + \frac{8}{x^3} - 4 = \frac{(-2+x)^2\left(4+4x+3x^2\right)}{2x^3} \geq 0$$

since $x \geq 0$. This ends the proof. Equality holds for $a = b = c$. □