Problema O145. Find all positive integers n for which

$$\left(1^4+\frac{1}{4}\right)\left(2^4+\frac{1}{4}\right)\cdots\left(n^4+\frac{1}{4}\right)$$

is the square of a rational number.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy

By using the Sophie Germain identity

$$\begin{aligned} a^4+4b^4 &= a^4+4a^2b^2+4b^4-4a^2b^2=\left(a^2+2b^2\right)^2-(2ab)^2 \\ &= \left(a^2+2b^2-2ab\right)\left(a^2+2b^2+2ab\right) \\ &= \left[(a-b)^2+b^2\right]\left[(a+b)^2+b^2\right] \end{aligned}$$

the given expression can be written in the following form

$$\begin{aligned} f(n) &= \left(1^4+\frac{1}{4}\right)\left(2^4+\frac{1}{4}\right)\cdots\left(n^4+\frac{1}{4}\right) \\ &= \frac{1}{4^n}\left(1^4+4\cdot 1^4\right)\left(1^4+4\cdot 2^4\right)\cdots\left(1^4+4\cdot n^4\right) \\ &= \frac{1}{4^n}\left(0^2+1^2\right)\left(2^2+1^2\right)\left(1^2+2^2\right)\left(3^2+2^2\right)\cdots\left((n-1)^2+n^2\right)\left((n+1)^2+n^2\right) \end{aligned}$$

from which easily follows that

$$f(n)=q_n^2\left(2n^2+2n+1\right) \tag{1}$$

where q_n is a rational number (this can be proved by induction).

From (1) it follows that $f(n)$ is the square of a rational number if and only if $2n^2+2n+1$ is a square. Therefore, it remains only to find the integer solutions to the diophantine equation

$$2n^2+2n+1=y^2 \qquad \Leftrightarrow \qquad (2n+1)^2-y^2=-1 \tag{2}$$

By setting $2n+1=x$ the equation (2) rewrites as

$$x^2-2y^2=-1 \tag{3}$$

In order to solve (3), we need the following result, whose proof can be found in a number theory book which deals with Pell's equation theory.

THEOREM. Every positive solution of the equation $x^2-2y^2=-1$ is given by (x_k, y_k), where x_k and y_k are the integers determined from

$$x_k+y_k\sqrt{2}=\left(1+\sqrt{2}\right)^{2k-1}$$

where k in a natural number. ■

Thanks to the above theorem, all the solutions (x_k, y_k) of equation (3) are given by

$$x_k = \frac{1}{2}\left[\left(1+\sqrt{2}\right)^{2k-1} + \left(1-\sqrt{2}\right)^{2k-1}\right] \tag{4}$$

$$y_k = \frac{1}{2}\left[\left(1+\sqrt{2}\right)^{2k-1} - \left(1-\sqrt{2}\right)^{2k-1}\right] \tag{5}$$

By using (4) and Mathematica, taking into account of $n = \frac{x-1}{2}$, we found the first twelve solutions of (3):

$$0, 3, 20, 119, 696, 4059, 23660, 137903, 803760, 4684659, 27304196, 159140519 \quad \square$$

Remark. The solutions (x_k, y_k) of equation (3) satisfy the recursive relation

$$\begin{aligned} x_{k+1} + y_{k+1}\sqrt{2} &= \left(1+\sqrt{2}\right)^{2k+1} = \left(1+\sqrt{2}\right)^{2k-1}\left(1+\sqrt{2}\right)^{2} \\ &= x_k + y_k\sqrt{2}\left(3+2\sqrt{2}\right) = 3x_k + 4y_k + (2x_k + 3y_k)\sqrt{2} \end{aligned}$$

Therefore we have

$$\begin{cases} x_{k+1} = 3x_k + 4y_k \\ y_{k+1} = 2x_k + 3y_k \end{cases} \qquad \Rightarrow \qquad x_{k+2} = 6x_{k+1} - x_k$$

with $x_1 = 1$ and $y_1 = 1$. Since $n = \frac{x-1}{2}$, a little calculation show that the solutions n_k of our problem satisfy the recursive relation

$$n_{k+2} = 6n_{k+1} - n_k + 2$$

with initial conditions $n_1 = 0$ and $n_2 = 3$. $\square$