

Problem 1209. Let R and r denote the radii of the circumcircle and the incircle of a triangle ABC with sides a, b, c and semi-perimeter s . Prove that

$$\frac{(s-a)^4}{c(s-b)} + \frac{(s-b)^4}{a(s-c)} + \frac{(s-c)^4}{b(s-a)} \geq \frac{3r}{4} \sqrt[3]{\frac{Rs^2}{2}}$$

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By using the well known inequalities

$$a^2 + b^2 + c^2 \geq ab + bc + ca, \quad 3(a^2 + b^2 + c^2) \geq (a + b + c)^2$$

as well as the Cauchy-Schwartz inequality in Engel form, we have

$$\begin{aligned} \mathbf{LHS} &\geq \frac{[(s-a)^2 + (s-b)^2 + (s-c)^2]^2}{c(s-b) + a(s-c) + b(s-a)} = \frac{[3s^2 - 2s(a+b+c) + a^2 + b^2 + c^2]^2}{(a+b+c)s - ab - bc - ca} = \\ &= \frac{(a^2 + b^2 + c^2 - s^2)^2}{2s^2 - ab - bc - ca} = \frac{(3a^2 + 3b^2 + 3c^2 - 2ab - 2ac - 2bc)^2}{8(a^2 + b^2 + c^2)} \geq \\ &\geq \frac{(a^2 + b^2 + c^2)^2}{8(a^2 + b^2 + c^2)} = \frac{a^2 + b^2 + c^2}{8} \geq \frac{(a+b+c)^2}{24} = \frac{s^2}{6} \geq \frac{3r}{4} \cdot \sqrt[3]{4Rs^2}. \end{aligned}$$

To justify the last step, from $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$, we get $8s^3 \geq 27 \cdot 4R\Delta$, or $2s^2 \geq 27Rr$. Now, since $R \geq 2r$, we have

$$\begin{aligned} 4s^4 &\geq 729 \cdot R^2 r^2 \geq 729 \cdot 2Rr^3 \Rightarrow \\ 8s^6 &\geq 729r^3 \cdot 4Rs^2 \Rightarrow 2s^2 \geq 9r \cdot \sqrt[3]{4Rs^2} \end{aligned}$$

whence, dividing by 12, we have the last step above, that is a refinement of the proposed inequality and where equality holds if and only if the triangle is equilateral. \square