Problema S109. Solve the system of equations

$$\sqrt{x}-\frac{1}{y}=\sqrt{y}-\frac{1}{z}=\sqrt{z}-\frac{1}{x}=\frac{7}{4}$$

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Adding the three equations we obtain:

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{21}{4} \qquad \Rightarrow$$

$$\left(\sqrt{x}-2\right)f(x)+\left(\sqrt{y}-2\right)f(y)+\left(\sqrt{z}-2\right)f(z)=0 \tag{1}$$

where

$$f(t)=\frac{2+\sqrt{t}+4t}{4t}>0, \qquad \forall t>0$$

Now $\sqrt{x}\geq 2$. In fact if $\sqrt{x}<2$ then

$$x<4 \quad \Rightarrow$$

$$\sqrt{z}=\frac{1}{x}+\frac{7}{4}>\frac{1}{4}+\frac{7}{4}=2 \quad \Rightarrow$$

$$\sqrt{y}=\frac{1}{z}+\frac{7}{4}<\frac{1}{4}+\frac{7}{4}=2 \quad \Rightarrow$$

$$\sqrt{x}=\frac{1}{y}+\frac{7}{4}>\frac{1}{4}+\frac{7}{4}=2$$

which is impossible. Similarly $\sqrt{y}<2$ or $\sqrt{z}<2$ leads to contradiction. Thus

$$\sqrt{x}-2\geq 0 \qquad , \qquad \sqrt{y}-2\geq 0 \qquad , \qquad \sqrt{z}-2\geq 0$$

Therefore, since $f(x)\geq 0$, $f(y)\geq 0$, $f(z)\geq 0$ from (1) follows that:

$$\sqrt{x}=\sqrt{y}=\sqrt{z}=2 \quad \Rightarrow \quad x=y=z=4$$

and we are done. $\square$