Problema S110. Let X be a point on the side BC of a triangle ABC. The parallel through X to AB meets CA at V and the parallel through X to AC meets AB at W. Let $D = BV \cap XW$ and $E = CW \cap XV$. Prove that DE is parallel to BC and

$$\frac{1}{DE} = \frac{1}{BX} + \frac{1}{CX}$$

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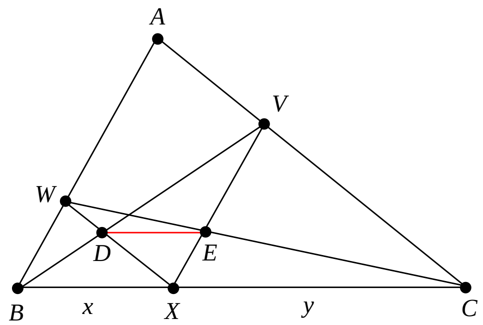
We denote $AB = c$, $BC = a$, $CA = b$, $BX = x$ and $XC = y$.



From the similarities $\triangle ACW \sim \triangle VCE$ and $\triangle WCB \sim \triangle ECX$ follows that:

$$AW : VE = WC : EC$$

$$WB : EX = WC : EC$$

and this implies:

$$AW : VE = WB : EX \quad \Longrightarrow \quad AW : WB = VE : EX \tag{1}$$

The similarity $\triangle ABV \sim \triangle WBD$ yields:

$$AB : WB = VB : DB \quad \Longrightarrow \quad AW : WB = VD : DB \tag{2}$$

From (1) and (2) we obtain:

$$VE : EX = VD : DB$$

Thus $\triangle VDE$ and $\triangle VBX$ are similar, so $DE \| BC$ and

$$DE : BX = VE : VX = AW = AB \quad \Longrightarrow \quad DE = \frac{AW \cdot BX}{AB}$$

Now we notice that:

$$BW : BX = AB : BC \quad \Longrightarrow \quad BW = \frac{AB \cdot BX}{BC} = \frac{cx}{a} \tag{3}$$

and

$$AW = AB - BW = c - \frac{cx}{a} = \frac{c}{a}(a - x) = \frac{cy}{a} \tag{4}$$

Finally, by replacing BW and AW with (3) and (4), we have:

$$DE = \frac{\frac{cy}{a} \cdot x}{c} = \frac{xy}{a} = \frac{xy}{x+y} \quad \Longrightarrow \quad \frac{1}{DE} = \frac{1}{x} + \frac{1}{y} = \frac{1}{BX} + \frac{1}{CX}$$

as desidered. $\square$