Problema S114. Consider triangle ABC with angle bisectors AA_1, BB_1, CC_1. Denote by U the intersection of AA_1 and B_1C_1. Let V be the projection from U onto BC. Let W be the intersection of the angle bisectors of $\angle BC_1V$ and $\angle CB_1V$. Prove that A, V, W are collinear.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ercole Suppa, Teramo, Italy

Denote as usual by a, b, c, R the sides BC, CA, AB and the circumradius of $\triangle ABC$, respectively. Let X, W_1 be the points where the the angle bisector of $\angle BC_1V$ intersects BC, AV and let Y, W_2 be the points where the the angle bisector of $\angle VB_1C$ intersects BC, AV (refere to FIGURE 1). In order to complete the proof is enough to show that $W_1 = W_2$.

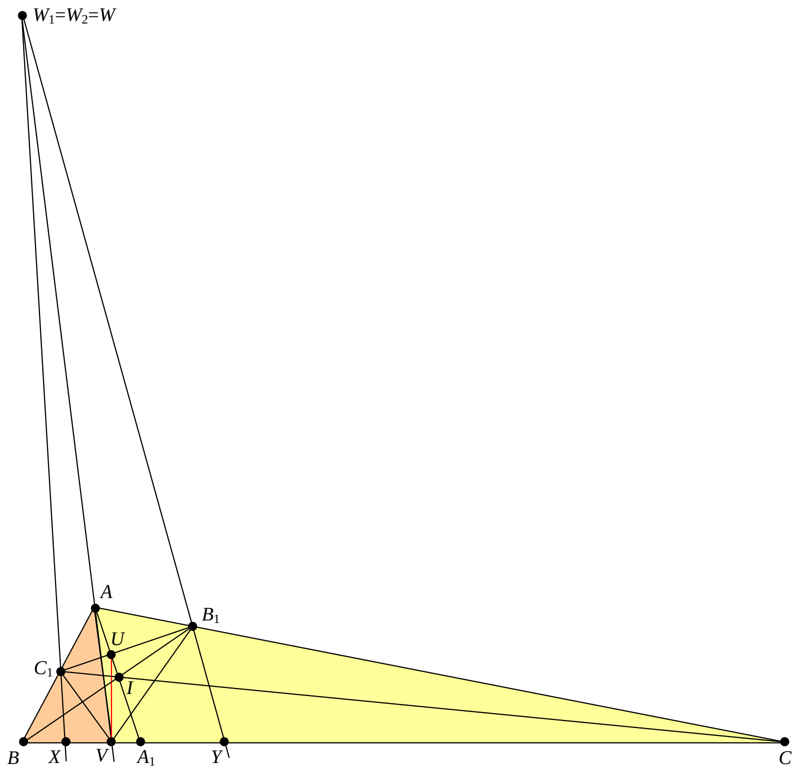

FIGURE 1

Applying Menelaus's theorem to triangle $\triangle ABV$ and the line XC_1W_1, and taking into account that $BX : XV = C_1B : C_1V$ (internal bisector theorem), we have:

$$\frac{AC_1}{C_1B} \cdot \frac{C_1B}{C_1V} \cdot \frac{VW_1}{W_1A} = 1 \tag{1}$$

In similar way applying Menelaus's theorem to triangle $\triangle AVC$ and the line YB_1W_2, and taking into account that $VY : YC = B_1V : B_1C$, we have:

$$\frac{AB_1}{B_1C} \cdot \frac{B_1C}{B_1V} \cdot \frac{VW_2}{W_2A} = 1 \tag{2}$$

From (1), (2) follows that:

$$\frac{AC_1}{C_1V} \cdot \frac{VW_1}{W_1A} = \frac{AB_1}{B_1V} \cdot \frac{VW_2}{W_2A} \tag{3}$$

In order to compute the ratio $\frac{C_1V}{B_1V}$, we draw the perpendiculars from C_1, B_1 to BC and denote with C_2, B_2 the projections of C_1, B_1, respectively.

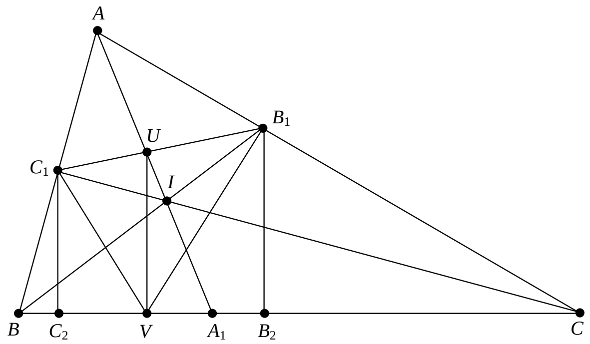

FIGURE 2

Using the Thales theorem and the well known relations $AC_1 = \frac{bc}{a+b}$, $BC_1 = \frac{ac}{a+b}$, $AB_1 = \frac{bc}{a+c}$, $CB_1 = \frac{ab}{a+c}$ we have:

$$\frac{C_1C_2}{B_1B_2} = \frac{BC_1 \sin B}{CB_1 \sin C} = \frac{\frac{ac}{a+b}\frac{2R}{a}}{\frac{ab}{a+c}\frac{2R}{b}} = \frac{a+c}{a+b} \tag{4}$$

$$\frac{C_2V}{VB_2} = \frac{C_1U}{UB_1} = \frac{AC_1}{AB_1} = \frac{\frac{bc}{a+b}}{\frac{bc}{a+c}} = \frac{a+c}{a+b} \tag{5}$$

From (4), (5) follows that:

$$\frac{C_1C_2}{B_1B_2} = \frac{C_2V}{VB_2} \tag{6}$$

Thus the right triangles $\triangle C_1C_2V$, $\triangle B_1B_2V$ are similar, so:

$$\frac{C_1V}{B_1V} = \frac{C_2V}{VB_2} = \frac{C_1U}{UB_1} = \frac{AC_1}{AB_1} \tag{7}$$

From (3), (7) follows that

$$\frac{VW_1}{W_1A} = \frac{VW_2}{W_2A} \qquad \Longrightarrow \qquad W_1 = W_2$$

and the proof is completed. $\square$