Problema S117. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{3abc}{2(ab+bc+ca)^2}\geq\frac{5}{a+b+c}$$

Proposed by Shamil Asgarli, Burnaby, Canada

Solution by Ercole Suppa, Teramo, Italy

The inequality between the harmonic and arithmetic means implies

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{9}{a+b+b+c+c+a}=\frac{9}{2(a+b+c)} \tag{1}$$

$$\frac{3abc}{a+b+c}=\frac{3}{\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}}\geq\frac{ab+bc+ca}{3} \tag{2}$$

Taking into account (1), (2) and the well-known inequality

$$(a+b+c)^2\geq 3(ab+bc+ca)$$

we have

$$\begin{aligned}
&\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{3abc}{2(ab+bc+ca)^2}\geq\\
&\geq\frac{9}{2(a+b+c)}+\frac{(a+b+c)(ab+bc+ca)}{3\cdot 2(ab+bc+ca)}\geq\\
&\geq\frac{9}{2(a+b+c)}+\frac{a+b+c}{2(a+b+c)^2}=\\
&=\frac{5}{a+b+c}
\end{aligned}$$

and the result is proven. We have equality if and only if $a=b=c$. □