Problema S119. Consider a point P inside a triangle ABC. Let AA_1, BB_1, CC_1 be cevians through P. The midpoint M of BC is different from A_1 and T is the intersection of AA_1 and B_1C_1. Prove that if the circumcircle of triangle BTC is tangent to the line B_1C_1, then $\angle BTM = \angle A_1TC$.

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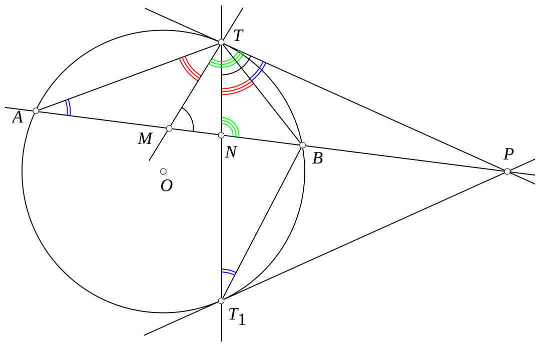
First we prove the following

LEMMA. From a point P, external to a circle (O), construct a tangent PT and a secant cutting the circle at A and B, as in figure. Let M be the midpoint of AB and let N the harmonic conjugate of P with respect to A, B. Prove that $\angle ATM = \angle NTB$.

Proof.

Let PT_1 be the second tangent to the circle and denote $X = TT_1 \cap AB$. It is well known that (A, X, B, P) is an harmonic quadruple, so $TT_1 \cap AB = N$.



Let $PA = u$, $PB = v$. Then $AB = u - v$, $PM = PB + \frac{1}{2}AB = \frac{u+v}{2}$ and

$$
\begin{aligned}
AN : NB = PA : PB \quad &\Longrightarrow \\
(AN + NB) : AN = (PA + PB) : PA \quad &\Longrightarrow \\
(u - v) : AN = (u + v) : u \quad &\Longrightarrow
\end{aligned}
$$

$$AN = \frac{u(u-v)}{u+v} \qquad , \qquad PN = PA - AN = \frac{2uv}{u+v}$$

By the tangent-secant theorem we have

$$PM \cdot PN = \frac{u+v}{2} \cdot \frac{2uv}{u+v} = uv = PA \cdot PB = PT^2 \quad \Longrightarrow \quad \frac{PT}{PN} = \frac{PM}{PT}$$

Thus, by S.A.S. similarity theorem, we have $\triangle TMP \sim \triangle NTP$ so

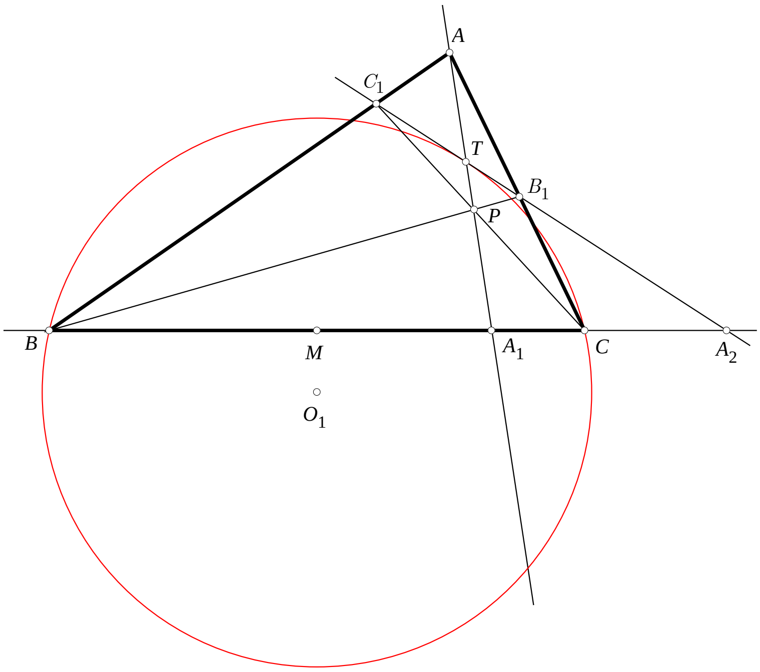
$$\angle TMP = \angle NTP = \angle NTB + \angle BTP \tag{1}$$

The exterior angle theorem yields:

$$\angle TMP = \angle ATM + \angle TAM \tag{2}$$

Since $\angle TAM = \angle TT_1B = \angle BTP$, (1) and (2) imply that $\angle ATM = \angle NTB$ and the lemma is proven. ■

Coming back to the problem and let $A_2 = B_1C_1 \cap BC$ as in figure below. By the LEMMA, in order to complete the proof, is enough to show that A_2 is the harmonic conjugate of A_1 with respect to B and C.



By the Ceva's and Menelao's theorems we have

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$$

$$\frac{BA_2}{A_2C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = -1$$

Therefore

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = -\frac{BA_2}{A_2C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} \qquad \Longrightarrow \qquad \frac{BA_1}{A_1C} = -\frac{BA_2}{A_2C}$$

i.e. A_1, A_2 are harmonic conjugates with respect to B, C so we are done. □