Problema S123. Prove that in any triangle with sidelengths a, b, c the following inequality holds:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}+\frac{(b+c-a)(c+a-b)(a+b-c)}{abc}\geq 7$$

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Our inequality is equivalent to:

$$bc(b+c)+ac(a+c)+ab(a+b)+(b+c-a)(c+a-b)(a+b-c)-7abc\geq 0 \quad (1)$$

Using the Ravi substitution $a=x+y$, $b=y+z$, $c=z+x$, $x,y,z\geq 0$, after some calculations, we can rewrite the inequality (1) in the following form:

$$x^3+y^3+z^3+3xyz\geq xy(x+y)+yz(y+z)+zx(z+x) \quad (2)$$

which is true because it is equivalent to the $r=1$ case of Schur's inequality:

$$x^r(x-y)(x-z)+y^r(y-x)(y-z)+z^r(z-x)(z-y)\geq 0$$

The proof is finished. Equality holds for $x=y=z$ or $x=y$, $z=0$ up to permutation, i.e. if the triangle is equilateral ($a=b=c$) or degenerate ($b=c$, $a=2b$ up to permutations). □