Problema S125. Find all pairs (p, q) of positive integers that satisfy

$$\left|\frac{p}{q} - \sqrt{2}\right| < \frac{1}{q^2}$$

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We need the following lemmas

Lemma 1. A pair (p, q) of positive integers, with $q \geq 2$, satisfies

$$\left|\frac{p}{q} - \sqrt{2}\right| < \frac{1}{q^2} \tag{1}$$

if and only if $\left|p^2 - 2q^2\right| \leq 2$ and $q \geq 2$.

Proof. ($\Rightarrow$) If (p, q) satisfies inequality (1) we have

$$\left|p - q\sqrt{2}\right| < \frac{1}{q} \qquad , \qquad \frac{p}{q} < \sqrt{2} + \frac{1}{q^2}$$

so

$$\left|p^2 - 2q^2\right| < \frac{1}{q}\left(p + q\sqrt{2}\right) = \frac{p}{q} + \sqrt{2} < 2\sqrt{2} + \frac{1}{q^2} \tag{2}$$

We now consider the following cases

- If $q = 2$ then $\left|p - 2\sqrt{2}\right| < 1/2$, hence

$$2\sqrt{2} - \frac{1}{2} < p < 2\sqrt{2} + \frac{1}{2} \quad \Rightarrow \quad p = 3 \quad \Rightarrow \quad \left|p^2 - 2q^2\right| \leq 2$$

- If $q \geq 3$, by (2) we have

$$\left|p^2 - 2q^2\right| \leq 2\sqrt{2} + \frac{1}{q^2} \leq 2\sqrt{2} + \frac{1}{9} \approx 2.9 \quad \Rightarrow$$

  and, since $p^2 - 2q^2$ is integer, $\left|p^2 - 2q^2\right| \leq 2$.

($\Leftarrow$) If (p, q) is a pair of positive integers, with $q \geq 2$, such that $\left|p^2 - 2q^2\right| \leq 2$, then we have

$$2q^2 - p^2 \leq \left|p^2 - 2q^2\right| \leq 2 \quad \Rightarrow \quad \frac{p^2}{q^2} \geq 2 - \frac{2}{q^2} \geq 1 \quad \Rightarrow \quad \frac{p}{q} \geq 1$$

Therefore

$$\left|\frac{p}{q} - \sqrt{2}\right| = \frac{\left|p^2 - 2q^2\right|}{q^2\left(\frac{p}{q} + \sqrt{2}\right)} \leq \frac{2}{q^2\left(1 + \sqrt{2}\right)} < \frac{1}{q^2}$$

and the lemma is proven. ∎

Lemma 2. Every positive solution of the equation $\left|x^2-2y^2\right|=1$ is given by (x_n, y_n), where $x_n$and y_n are the integers determined from

$$x_n + y_n\sqrt{2} = \left(1+\sqrt{2}\right)^n \quad , \quad n \in \mathbb{N}$$

Furthermore if n in odd $x_n^2 - 2y_n^2 = -1$, whereas if n in even $x_n^2 - 2y_n^2 = 1$.

Proof. It can be found in a number theory book which deals with Pell's equation theory. ■

Now we will solve the problem.

If $q = 1$ we find the pairs $(1, 1)$ and $(2, 1)$.

If $q \geq 2$, due to the Lemma 1, it is enough to find all pairs (p, q) of positive integers that satisfy $\left|p^2-2q^2\right| \leq 2$. Since $p^2 - 2q^2 \neq 0$ we must solve the equations $\left|p^2-2q^2\right| = 1$ and $\left|p^2-2q^2\right| = 2$.

By using the Lemma 2, the solutions of $\left|p^2-2q^2\right| = 1$ are the pairs (p_n, q_n) such that

$$p_n + q_n\sqrt{2} = \left(1+\sqrt{2}\right)^n \quad , \quad n \in \mathbb{N}$$

Since

$$\begin{aligned} p_{n+1} + q_{n+1}\sqrt{2} &= \left(1+\sqrt{2}\right)^{n+1} = \left(1+\sqrt{2}\right)^n \cdot \left(1+\sqrt{2}\right) = \\ &= \left(p_n + q_n\sqrt{2}\right) \cdot \left(1+\sqrt{2}\right) = p_n + 2q_n + (p_n + q_n)\sqrt{2} \end{aligned}$$

the solutions (p_n, q_n) are given by the following recurrence

$$A \; : \; \begin{cases} p_{n+1} = p_n + 2q_n \\ q_{n+1} = p_n + q_n \end{cases} \quad , \quad n \in \mathbb{N} \tag{3}$$

with initial conditions $p_1 = 1$ and $q_1 = 1$.

In order to solve the equation $\left|p^2-2q^2\right| = 2$, let us observe that p must be even. Thus, by setting $q = u$ and $p = 2v$ the equation turns into

$$\left|u^2-2v^2\right| = 1$$

whose solutions (u_n, v_n) are given by the recurrences $u_{n+1} = u_n + 2v_n$, $v_{n+1} = u_n + v_n$ $(n \in \mathbb{N})$, with $u_1 = 1$ and $v_1 = 1$.

Then, after an easy calculation, we find that the solutions of $\left|p^2-2q^2\right| = 2$ are

$$B \; : \; \begin{cases} p_{n+1} = p_n + 2q_n \\ q_{n+1} = p_n + q_n \end{cases} \quad , \quad n \in \mathbb{N} \tag{4}$$

with initial conditions $p_1 = 2$ and $q_1 = 1$.

Finally we have proved that the solutions of the proposed inequality can be obtained by means of the recurrences (3) and (4) which, obviously, include also the pairs $(1, 1)$ and $(2, 1)$.

By means of MATHEMATICA we have listed the first thirteen solutions given by A and B:

$$A=\{\frac{1}{1},\frac{3}{2},\frac{7}{5},\frac{17}{12},\frac{41}{29},\frac{99}{70},\frac{239}{169},\frac{577}{408},\frac{1393}{985},\frac{3363}{2378},\frac{8119}{5741},\frac{19601}{13860},\frac{47321}{33461},\dots\}$$

$$B=\{\frac{2}{1},\frac{4}{3},\frac{10}{7},\frac{24}{17},\frac{58}{41},\frac{140}{99},\frac{338}{239},\frac{816}{577},\frac{1970}{1393},\frac{4756}{3363},\frac{11482}{8119},\frac{27720}{19601},\frac{66922}{47321},\dots\}$$

$\square$