**Problema S137.** Let $ABCD$ be a cyclic quadrilateral and let $U = AB \cap CD$ and $V = BC \cap AD$. The line that passes through $V$ and is perpendicular to the angle bisector of angle $\angle AUD$ intersects $UA$ and $UD$ at $X$ and $Y$, respectively. Prove that

$$AX \cdot DY = BX \cdot CY$$

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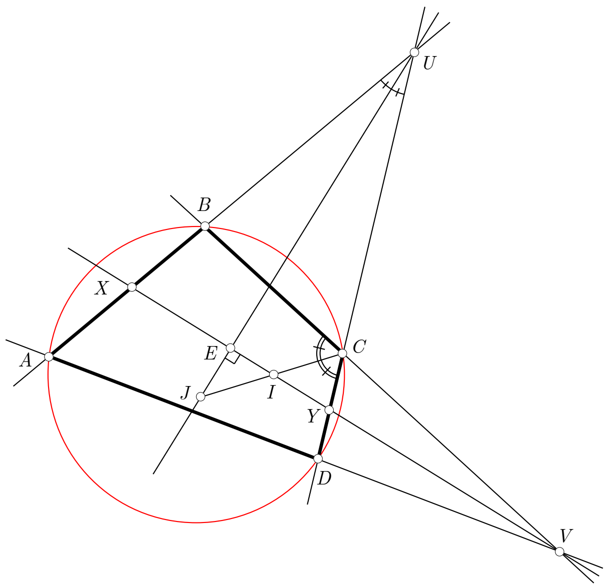
Let $E$, $J$ be the intersection points of the line $XY$ with the angle bisectors of angles $\angle AUD$, $\angle BCD$ respectively and let $I = XY \cap CJ$, as shown in figure.




*Claim:* The line $XY$ is the angle bisector of angle $\angle AVB$.

*Proof.* Let us denote with $A$, $B$, $C$, $D$ the measures of internal angles of quadrilateral $ABCD$. We have

$$\angle AVB = 180^\circ - (A + B) \qquad , \qquad \angle AUD = 180^\circ - (A + D) \tag{1}$$

and

$$\begin{aligned} \angle CIV = \angle EIJ = 90^\circ - \angle EJI = 90^\circ - \angle UJC = \\ = 90^\circ - (180^\circ - \angle JUC - \angle JCU) = \\ = \angle JUC + \angle JCU - 90^\circ = \\ = \frac{180^\circ - (A + D)}{2} + \left(180^\circ - \frac{C}{2}\right) - 90^\circ = \\ = \frac{C}{2} - \frac{D}{2} + 90^\circ - \frac{C}{2} = 90^\circ - \frac{D}{2} \end{aligned} \tag{2}$$

From (1), (2) it follows that

$$\begin{aligned}\angle BVX = \angle CVI = 180^\circ - \angle CIV - \angle ICV = \\ = 180^\circ - \left(90^\circ - \frac{D}{2}\right) - \left(180^\circ - \frac{C}{2}\right) = \\ = \frac{D}{2} + \frac{C}{2} - 90^\circ = 90^\circ - \frac{A+B}{2} = \frac{1}{2}\angle AVB\end{aligned}$$

and the claim is proven. ■

Coming back to the problem, applying the internal bisector theorem to the triangles $\triangle VCD$ and $\triangle VAB$ we obtain

$$CY : YD = VC : VD \qquad \Rightarrow \qquad VC = \frac{CY}{YD} \cdot VD \tag{3}$$

$$BX : XA = VB : VA \qquad \Rightarrow \qquad VB = \frac{BX}{XA} \cdot VA \tag{4}$$

From (3), (4), taking into account that $VC \cdot VB = VD \cdot VA$, we get

$$\frac{CY}{YD} \cdot VD \cdot \frac{BX}{XA} \cdot VA = VD \cdot VA \qquad \Rightarrow \qquad AX \cdot DY = BX \cdot CY$$

and the proof is complete. □