Problema S138. Let a, b, c be positive real numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$. Prove that

$$8\left(a^2+b^2+c^2\right) \geq 3(a+b)(b+c)(c+a)$$

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Solution by Ercole Suppa, Teramo, Italy

By AM-GM inequality we have

$$3(a+b)(b+c)(c+a) \leq 3\left(\frac{2a+2b+2c}{3}\right)^3 = \frac{8}{9}(a+b+c)^3 \tag{1}$$

On the other hand, taking $x_i = a^{1/3}$, $y_i = b^{2/3}$, $p = \frac{3}{2}$, $q = 3$, in the Hölder's inequality

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {y_i}^q\right)^{\frac{1}{q}}$$

we get

$$a+b+c = a^{\frac{1}{3}}a^{\frac{2}{3}} + b^{\frac{1}{3}}b^{\frac{2}{3}} + c^{\frac{1}{3}}c^{\frac{2}{3}} \leq \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^{\frac{2}{3}} \left(a^2+b^2+c^2\right)^{\frac{1}{3}} \quad \Rightarrow$$

$$(a+b+c)^3 \leq \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \left(a^2+b^2+c^2\right) \tag{2}$$

Since $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$ from (2) it follows that

$$(a+b+c)^3 \leq 9\left(a^2+b^2+c^2\right) \tag{3}$$

From (1) and (3) we get the desired inequality.

The equality holds if $a=b=c=1$. $\square$