**Problema S140.** Let $a$, $b$, $c$ be integers. Prove that

$$\sum_{cyc}(a-b)\left(a^2+b^2-c^2\right)c^2$$

is divisible by $(a+b+c)^2$.

*Proposed by Dorin Andrica, "Babes-Bolyai" University, Cluj-Napoca, Romania*

*Solution by Ercole Suppa, Teramo, Italy*

Clearly we have

$$\begin{aligned}
&\sum_{cyc}(a-b)\left(a^2+b^2-c^2\right)c^2 = \\
&= \sum_{cyc} a^3c^2 + \sum_{cyc} ab^2c^2 - \sum_{cyc} ac^4 - \sum_{cyc} a^2bc^2 - \sum_{cyc} b^3c^2 + \sum_{cyc} bc^4 = \\
&= \sum_{cyc} a^3c^2 - \sum_{cyc} ac^4 - \sum_{cyc} b^3c^2 + \sum_{cyc} bc^4 = \\
&= \sum_{cyc} a^3\left(c^2-b^2\right) + \sum_{cyc} a^4(c-b) = \\
&= \sum_{cyc} a^3(c-b)(a+b+c) = (a+b+c)\sum_{cyc} a^3(c-b) \qquad (1)
\end{aligned}$$

so the given sum is divisible by $a+b+c$.

Let us rewrite the expression $\sum_{cyc} a^3(c-b)$ in the following form

$$\begin{aligned}
\sum_{cyc} a^3(c-b) &= a^3(c-b)+b^3(a-c)+c^3(b-a) = \\
&= (b-a)\left[c^3-\left(a^2+b^2+ab\right)c-ab(a+b)\right] \qquad (2)
\end{aligned}$$

Now, observe that the roots $\alpha$, $\beta$, $\gamma$ of the polynomial

$$P(c)=c^3-\left(a^2+b^2+ab\right)c-ab(a+b)$$

satisfy the Viète's formulas

$$\alpha+\beta+\gamma=0 \quad , \quad \alpha\beta+\beta\gamma+\gamma\alpha=-a^2-b^2-ab \quad , \quad \alpha\beta\gamma=-ab(a+b)$$

An easy computation show that $\alpha=a$, $\beta=b$, $\gamma=-a-b$, hence

$$P(c)=(c-a)(c-b)(c+a+b) \qquad (3)$$

Finally, from (1),(2) and (3) it follows that

$$\sum_{cyc}(a-b)\left(a^2+b^2-c^2\right)c^2=(b-a)(c-a)(c-b)(a+b+c)^2$$

and the proof is finished. $\square$