Problema S144. Let $ABCD$ be a quadrilateral. We consider the reflection of the lines AB, BC, CD, DA on the respective midpoints of the opposite sides CD, DA, AB, BC. Prove that these four lines bound a quadrilateral $A'B'C'D'$ homothetic with $ABCD$ and find the ratio and center of the homothety.

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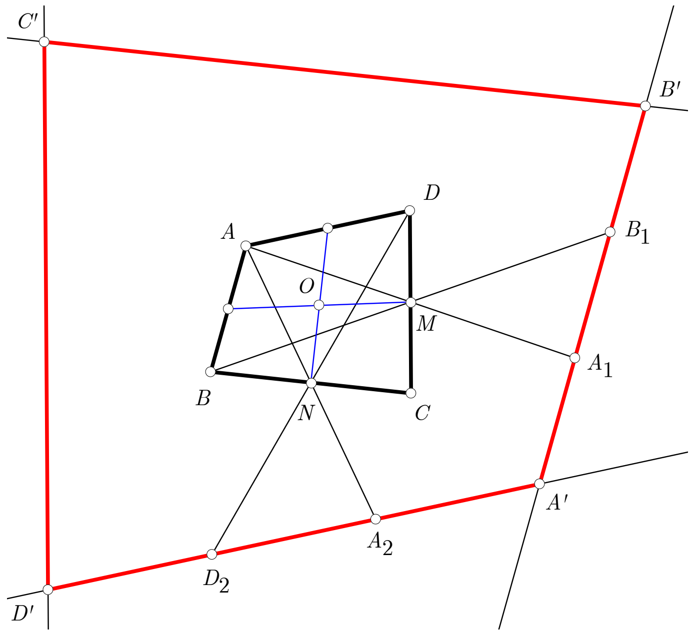
Consider a system of coordinates with origin in the centroid O of $ABCD$ and denote by $\overrightarrow{X}$ the vector from O to X. Let A_1, B_1 be the reflections of A, B on the midpoint M of CD and let A_2, D_2 be the reflections of A, D on the midpoint N of BC, as shown in figure.



We clearly have $\overrightarrow{M} = \frac{\overrightarrow{C}+\overrightarrow{D}}{2}$, $\overrightarrow{N} = \frac{\overrightarrow{B}+\overrightarrow{C}}{2}$, hence

$$\overrightarrow{A_1} = \overrightarrow{C} + \overrightarrow{D} - \overrightarrow{A}, \qquad \overrightarrow{B_1} = \overrightarrow{C} + \overrightarrow{D} - \overrightarrow{B} \tag{1}$$

$$\overrightarrow{A_2} = \overrightarrow{B} + \overrightarrow{C} - \overrightarrow{A}, \qquad \overrightarrow{D_2} = \overrightarrow{B} + \overrightarrow{C} - \overrightarrow{D} \tag{2}$$

Since A' lies on the lines A_1B_1 and A_2D_2 there are suitable real numbers t, u such that

$$\overrightarrow{A'} = \overrightarrow{A_1} + t\left(\overrightarrow{B_1} - \overrightarrow{A_1}\right) = \overrightarrow{A_2} + u\left(\overrightarrow{D_2} - \overrightarrow{A_2}\right)$$

and using (1) and (2) we get

$$\overrightarrow{C} + \overrightarrow{D} - \overrightarrow{A} + t\left(\overrightarrow{A} - \overrightarrow{B}\right) = \overrightarrow{B} + \overrightarrow{C} - \overrightarrow{A} + u\left(\overrightarrow{A} - \overrightarrow{D}\right) \qquad \Rightarrow$$

$$(t-u)\overrightarrow{A} - (t+1)\overrightarrow{B} + (u+1)\overrightarrow{D} = 0$$

Since the vectors $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{D}$ are linearly independent we obtain

$$t = u = -1 \qquad \Rightarrow \qquad \overrightarrow{A'} = \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D} - 2\overrightarrow{A} \tag{3}$$

From (3), taking into account that $\overrightarrow{O} = \frac{1}{4}\left(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D}\right)$, it follows that

$$\overrightarrow{A'} + 3 \cdot \overrightarrow{A} = 4 \cdot \overrightarrow{O} \qquad \Leftrightarrow \qquad \overrightarrow{OA'} = -3 \cdot \overrightarrow{OA}$$

and this implies that A' is the image of A under the homotety of center O and ratio $k = -3$. In similar way we can prove that B', C', D' are respectively the correspondents of B, C, D in the homotety with center O and ratio $k = -3$.

Therefore the quadrilateral $A'B'C'D'$ is the image of $ABCD$ under the homotety of center O and ratio $k = -3$. $\square$