Problema S146. Let m_a, m_b, m_c be the medians, k_a, k_b, k_c the symmedians, r the inradius, and R the circumradius of a triangle ABC. Prove that

$$\frac{3R}{2r} \geq \frac{m_a}{k_a} + \frac{m_b}{k_b} + \frac{m_c}{k_c} \geq 3$$

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By using the well known formulas

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \qquad , \qquad k_a = \frac{bc}{b^2 + c^2}\sqrt{2b^2 + 2c^2 - a^2}$$

the given inequality equivals to

$$\frac{3R}{r} \geq \frac{a^2 + b^2}{ab} + \frac{b^2 + c^2}{bc} + \frac{c^2 + a^2}{ca} \geq 6 \tag{1}$$

The right-hand side of (1) is obvious since

$$\frac{a^2 + b^2}{ab} \geq 2 \quad , \quad \frac{b^2 + c^2}{bc} \geq 2 \quad , \quad \frac{c^2 + a^2}{ca} \geq 2$$

For the left-hand side, we prove at first following inequality

$$\frac{R}{r} \geq \frac{a^2 + b^2}{ab} \tag{2}$$

By using Ravi's transformations $a = y + z$, $b = x + z$, $c = x + y$, the above inequality rewrites as

$$\begin{aligned}
\frac{(x+y)(y+z)(z+x)}{4xyz} &\geq \frac{y+z}{x+z} + \frac{x+z}{y+z} && \Leftrightarrow \\
\frac{x+y}{4xyz} &\geq \frac{1}{(x+z)^2} + \frac{1}{(y+z)^2} && \Leftrightarrow \\
\frac{1}{4yz} + \frac{1}{4xz} &\geq \frac{1}{(x+z)^2} + \frac{1}{(y+z)^2}
\end{aligned}$$

which is true since $(x + z)^2 \geq 4xz$ and $(y + z)^2 \geq 4yz$.

Summing up (2) and similar cyclic results we get the desired inequality. Moreover, the equality holds only in the case $a = b = c$. $\square$