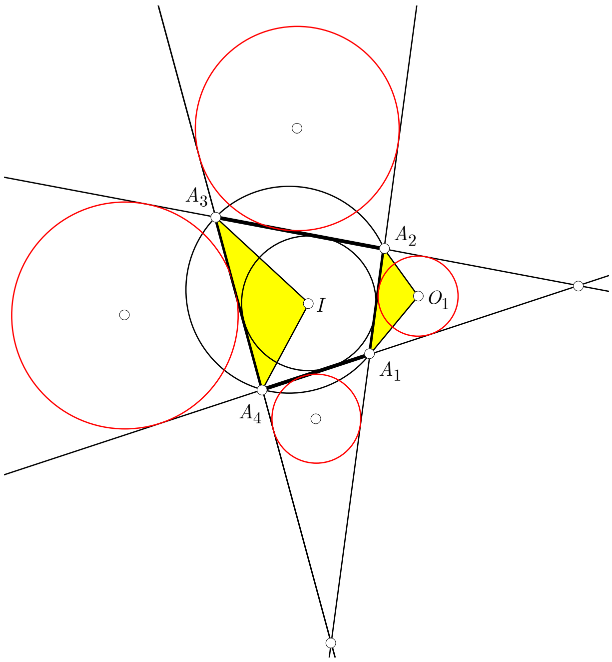
Problema S150. Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle $C(O,R)$ and circumscribed about a circle $omega(I,r)$. Denote by R_i the radius of the circle tangent to A_iA_{i+1} and tangent to the extensions of the sides $A_{i-1}A_i$ and $A_{i+1}A_{i+2}$. Prove that the sum $R_1+R_2+R_3+R_4$ does not depend on the position of points A_1, A_2, A_3, A_4.

Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA

Solution by Ercole Suppa, Teramo, Italy

The claimed result is wrong, as we can verify with a dynamic geometry software. Anyway, we can express the sum $R_1+R_2+R_3+R_4$ by means of r and the sidelengths of the quadrilateral $A_1A_2A_3A_4$.



Let us denote $A_1A_2=a$, $A_2A_3=b$, $A_3A_4=c$, $A_4A_1=d$ and let O_i be the center of the circle tangent to A_iA_{i+1} and tangent to the extensions of the sides $A_{i-1}A_i$ and $A_{i+1}A_{i+2}$.

Since $A_1A_2A_3A_4$ is cyclic we have $\angle O_1A_2A_1 = \angle A_3A_4I$ and $\angle O_1A_1A_2 = \angle A_4A_3I$. Thus the triangles $\triangle O_1A_1A_2$ and $\triangle O_1A_3A_4$ are similar, so

$$\frac{R_1}{a}=\frac{r}{c} \qquad \Rightarrow \qquad R_1 = r\cdot\frac{a}{c} \tag{1}$$

Summing up (1) and similar cyclic results we get

$$R_1+R_2+R_3+R_4 = r\left(\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}\right)$$

as desired. $\square$