Problema S164. Let $ABCD$ be a cyclic quadrilateral whose diagonals are perpendicular to each other. For a point P on its circumscribed circle denote by ℓ_P he line tangent to the circle at P. Let $U = \ell_A \cap \ell_B$, $V = \ell_B \cap \ell_C$, $W = \ell_C \cap \ell_D$, $K = \ell_D \cap \ell_A$. Prove that $UVWK$ is a cyclic quadrilateral.

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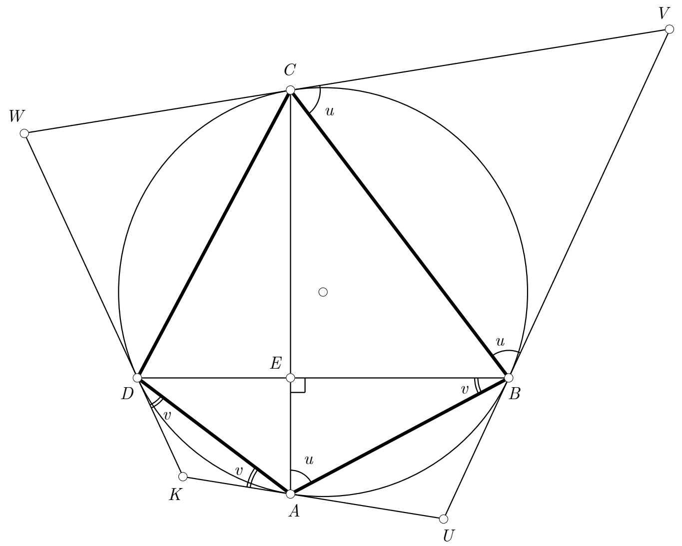
Let $E = AC \cap BD$, $\angle VCB = \angle VBC = u$, $\angle KDA = \angle KAD = v$, as showed in the following figure



Since $\angle CAB = \angle VCB = u$, $\angle ABD = \angle KAD = v$ and $u + v = \angle EAB + \angle EBA = 90^\circ$ we get

$$\begin{aligned}\angle WVU + \angle WKU &= \angle CVB + \angle AKD = (180^\circ - 2u) + (180^\circ - 2v)\\ &= 360^\circ - 2(u + v) = 360^\circ - 180^\circ = 180^\circ\end{aligned}$$

Therefore the quadrilateral $UVWK$ is cyclic and we are done. $\square$