

Problema S165. Let I be the incenter of triangle ABC . Prove that

$$AI \cdot BI \cdot CI \geq 8r^3$$

Proposed by Dorin Andrica, "Babes-Bolyai University", Cluj-Napoca, Romania

Solution by Ercole Suppa, Teramo, Italy

Let s , R , Δ denote the semiperimeter, the circumradius and the area of triangle ABC . Using the Briggs's formulas

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

we have

$$AI \cdot BI \cdot CI = \frac{r}{\sin \frac{A}{2}} \cdot \frac{r}{\sin \frac{B}{2}} \cdot \frac{r}{\sin \frac{C}{2}} = \frac{r^3 abc}{(s-a)(s-b)(s-c)}$$

Therefore, employing the well known formulas $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, $rs = \Delta$, $abc = 4R\Delta$ and the Euler inequality $R \geq 2r$, we get the desired result

$$AI \cdot BI \cdot CI = \frac{r^3 sabc}{\Delta^2} = \frac{r^3 s \cdot 4R\Delta}{\Delta^2} = 4r^2 R \geq 8r^3$$

□