Problema J163. Let a, b, c be positive real numbers. Prove that

$$\sum_{cyc} \frac{a^2b^2(b-c)}{a+b} \geq 0$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Ercole Suppa, Teramo, Italy

Since

$$\sum_{cyc} \frac{a^2b^2(b-c)}{a+b} \geq \sum_{cyc} \frac{a^2b^2(b-c)}{a+b+c}$$

it suffices to prove that

$$a^2b^2(b-c) + b^2c^2(c-a) + c^2a^2(a-b) \geq 0 \tag{1}$$

In order to prove (1) we consider the following cases

(i) The first case $a \leq b \leq c$. We clearly have

$$\begin{aligned} & a^2b^2(b-c) + b^2c^2(c-a) + c^2a^2(a-b) = \\ =& a^2b^2(b-c) + b^2c^2(c-b) + b^2c^2(b-a) + c^2a^2(a-b) = \\ =& b^2(c-b)\left(c^2-a^2\right) + c^2(b-a)\left(b^2-a^2\right) \geq 0 \end{aligned}$$

(ii) The second case $c \leq b \leq a$. We clearly have

$$\begin{aligned} & a^2b^2(b-c) + b^2c^2(c-a) + c^2a^2(a-b) = \\ =& a^2b^2(b-c) + b^2c^2(c-a) + c^2a^2(a-c) + c^2a^2(c-b) = \\ =& a^2(b-c)\left(b^2-c^2\right) + c^2(a-c)\left(a^2-b^2\right) \geq 0 \end{aligned}$$

This ends the proof. The equality holds for $a = b = c$. $\square$