

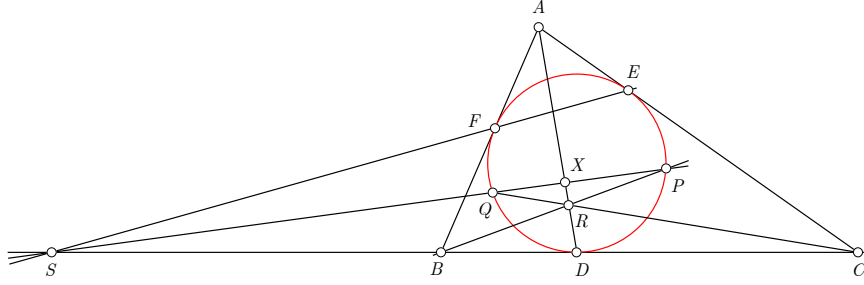
Problema S218. Let ABC be a triangle with incircle \mathcal{C} and incenter I . Let D, E, F be the tangency points of \mathcal{C} with the sides BC, CA , and AB , respectively, and furthermore, let S be the intersection of BC and EF . Let P, Q be the intersection points of SI with \mathcal{C} such that P, Q lie on the small arcs DE and FD respectively. Prove that the lines AD, BP, CQ are concurrent.

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We will prove that the result is true if the point I is replaced with any point X on the segment AD , i.e.:

Let P, Q be the intersection points of SX with \mathcal{C} such that P, Q lie on the small arcs DE and FD respectively. Prove that the lines AD, BP, CQ are concurrent (see figure).



Let $a = BC, b = CA, c = AB$ and let s be the semiperimeter of $\triangle ABC$. It is well known that $AF = FE = s - a, BD = BF = s - b, CD = CE = s - c$, so

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{s-a}{s-b} \cdot \frac{s-b}{s-c} \cdot \frac{s-c}{s-a} = 1 \quad (1)$$

Applying Menealaus theorem to $\triangle ABC$ with transversal EFS we have

$$\frac{AF}{FB} \cdot \frac{BS}{SC} \cdot \frac{CE}{EA} = 1 \quad (2)$$

From (1), (2) it follows that

$$\frac{AF}{FB} \cdot \frac{DB}{DC} \cdot \frac{CE}{EA} = \frac{AF}{FB} \cdot \frac{BS}{SC} \cdot \frac{CE}{EA} \quad \Leftrightarrow \quad \frac{BD}{DC} = -\frac{BS}{SC}$$

i.e. the division (S, D, B, C) is harmonic.

Let $R = AD \cap CQ$ and let $P' = BR \cap SX$. Since $(SDBC) = -1$ it follows that $R(SDBC)$ is an harmonic bundle, so

$$(SXP'Q) = -1 \quad \Rightarrow \quad (SXQP') = -1 \quad (3)$$

Since S lies on the polar of A wrt \mathcal{C} , A lies on the polar of S wrt \mathcal{C} . Therefore AD is the polar of S with respect to \mathcal{C} , so

$$(SXQP) = -1 \quad (4)$$

Taking into account of (3) and (4), the uniqueness of fourth harmonic yields $P' = P$ and this prove that AD, BP, CQ are concurrent. \square