Problema S346. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{15}{(a+b+c)^2}\geq\frac{6}{ab+bc+ca}$$

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First solution by Ercole Suppa, Teramo, Italy

Putting $p=a+b+c>0$, $q=ab+bc+ca>0$, $r=abc>0$ and clearing the denominators, after some algebra, the given inequality become equivalent to

$$p^6q-8p^4q^2+16p^2q^3+16p^3qr-30pq^2r-6p^2r^2+15qr^2\geq 0 \qquad \Leftrightarrow$$

$$\left(p^6q+16p^2q^3-8p^4q^2\right)+\left(10p^3qr-30pq^2r\right)+\left(6p^3qr-6p^2r^2\right)+15qr^2\geq 0 \tag{1}$$

From AM-GM inequality it follows that

$$p^6q+16p^2q^3-8p^4q^2\geq 2\sqrt{16p^8q^4}-8p^4q^2=0 \tag{2}$$

From the well know inequality $(a+b+c)^2\geq 3(ab+bc+ca)$ we have that $p^2-3q\geq 0$, so

$$10p^3qr-30pq^2r\geq 0 \tag{3}$$

From AM-GM inequality it follows that

$$pq=(a+b+c)(ab+bc+ca)\geq 3\sqrt[3]{abc}\cdot 3\sqrt[3]{a^2b^2c^2}=9abc=9r\geq r \qquad \Leftrightarrow$$
$$6p^3qr-6p^2r^2\geq 0 \tag{4}$$

Summing (2), (3), (4) and the obvious inequality $15qr^2\geq 0$ we get (1) which is exactly what we want. □

Second solution by Ercole Suppa, Teramo, Italy

After some boring algebra we see that

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{15}{(a+b+c)^2}-\frac{6}{ab+bc+ca}=A+B+C+D$$

where

$$A=\frac{(a+b+c)^2(ab+bc+ca)\left(a^2+b^2+c^2-2ab-2bc-2ca\right)^2}{(a+b)^2(b+c)^2(c+a)^2(a+b+c)^2(ab+ac+bc)}$$
$$B=\frac{10abc(a+b+c)(ab+bc+ca)\left(a^2+b^2+c^2-ab-bc-ca\right)}{(a+b)^2(b+c)^2(c+a)^2(a+b+c)^2(ab+ac+bc)}$$
$$C=\frac{6abc(a+b)(a+c)(b+c)(a+b+c)^2}{(a+b)^2(a+c)^2(b+c)^2(a+b+c)^2(ab+bc+ca)}$$
$$D=\frac{15a^2b^2c^2(ab+bc+ca)}{(a+b)^2(b+c)^2(c+a)^2(a+b+c)^2(ab+ac+bc)}$$

Since A,B,C,D are (obviously) positive numbers, we obtain

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{15}{(a+b+c)^2}-\frac{6}{ab+bc+ca}\geq 0$$

and the equality holds if and only if $a=0$, $b=c$ and its cyclic permutations. □