

Problema U115. Let $a_n = 2 - \frac{1}{n^2 + \sqrt{n^4 + \frac{1}{4}}}$, $n = 1, 2, \dots$. Prove that

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_{119}}$$

is an integer.

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The nested radical formula leads to

$$\begin{aligned}\sqrt{a_n} &= \sqrt{2 + 4n^2 - 4\sqrt{n^4 + \frac{1}{4}}} = \\ &= \sqrt{2 + 4n^2 - \sqrt{16n^4 + 4}} = \\ &= \sqrt{\frac{2 + 4n^2 + 4n}{2}} - \sqrt{\frac{2 + 4n^2 - 4n}{2}} = \\ &= \sqrt{2n^2 + 2n + 1} - \sqrt{2n^2 - 2n + 1} = \\ &= \sqrt{(n+1)^2 + n^2} - \sqrt{n^2 + (n-1)^2}\end{aligned}$$

By setting $b_n = \sqrt{n^2 + (n-1)^2}$ we get $a_n = b_{n+1} - b_n$. Thus

$$\begin{aligned}\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_{119}} &= (b_2 - b_1) + (b_3 - b_2) + \cdots + (b_{120} - b_{119}) = \\ &= b_{120} - b_1 = \sqrt{120^2 + 119^2} - 1 = 168\end{aligned}$$

and we are done. □