Problema U116. Let G be a K_4 complete graph without an edge. Find the number of closed walks of length n in G.

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Let us call $n-$walk a walk of length n in G. We use the following results

LEMMA 1. If $A = (a_{ij})$ is the adjacency matrix of a graph G, then the (i, j)-entry $a_{ij}^{(n)}$ of the matrix A^n is equal to the number of $n-$walks that originate at vertex i and terminate at vertex j.

Proof.

We proceed by induction. The lemma is trivially true for $n = 1$. For inductive hypothesis, assume that the lemma holds for for $n-1$, i.e. assume that the (i, j) entry of the matrix $A^{n-1} = \left(a_{ij}^{(n-1)}\right)$ represent the number of the walks of length $n-1$ from vertex i to vertex j. The (i, j) entry of A^n is given by

$$a_{ij}^{(n)} = \sum_{k=1}^{n} a_{ik}^{(n-1)} \cdot a_{kj} \tag{1}$$

In (1), $a_{kj} = 1$ or 0 depending on whether or not there is a walk from k to j. Thus a term of sum (1) is non zero if and only if there is a $n-$walk from i to j, whose last edge is from k to j. If the term is not zero, its value equals the number of such edge sequences from i to j via k, where $1 \le k \le n$. Therefore (1) is equal to the number of all possible $n-$walks from i to j and the lemma is proven. ■

LEMMA 2. Let A be a square matrix, let λ be an eigenvalue of A and let $n \ge 0$ be an integer. Then λ^n is an eigenvalue of A^n.

Proof.

Let $x \neq 0$ be an eigenvector of A corresponding to the eigenvalue λ. We proceed by induction on n. Obviously for $n = 0$ we have

$$A^0 x = Ix = x = \lambda^0 x$$

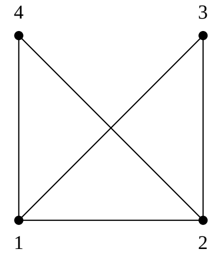
so λ^0 is an eigenvalue of A^0. If we assume for inductive hypothesis that the lemma is true for n, then we have

$$A^{n+1}x = (A^n A)\, x = A^n(\lambda x) = \lambda\,(A^n x) = \lambda\,(\lambda^n x) = \lambda^{n+1}x$$

So x is an eigenvector of A^{n+1} for λ^{n+1}, and induction tell us the theorem is true for all $n \geq 0$. $\blacksquare$

Coming back to the problem, let us determine the adjacency matrix A of our graph G

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$



In order to calculate the eigenvalues of A, let us solve the charateristic equation

$$\begin{vmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 0 \\ 1 & 1 & 0 & -\lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad \lambda^4 - 5\lambda^2 - 4\lambda = 0$$

from which we obtain

$$\lambda_1 = \frac{1-\sqrt{17}}{2} \quad , \quad \lambda_2 = -1 \quad , \quad \lambda_3 = 0 \quad , \quad \lambda_4 = \frac{1+\sqrt{17}}{2}$$

So, by LEMMA 2, the eigenvalues of $A^n = \left(a_{ij}^{(n)}\right)$ are

$$\lambda_1^n = \left(\frac{1-\sqrt{17}}{2}\right)^n \quad , \quad \lambda_2^n = (-1)^n \quad , \quad \lambda_3^n = 0 \quad , \quad \lambda_4^n = \left(\frac{1+\sqrt{17}}{2}\right)^n$$

Finally, by using the LEMMA 1, we have that the number c_n of closed $n-$walks in G is given by

$$\begin{aligned} c_n &= a_{11}^{(n)} + a_{22}^{(n)} + a_{33}^{(n)} + a_{44}^{(n)} = \text{Trace}\,(A^n) = \\ &= \lambda_1^n + \lambda_2^n + \lambda_3^n + \lambda_4^n = \\ &= (-1)^n + \left(\frac{1-\sqrt{17}}{2}\right)^n + \left(\frac{1+\sqrt{17}}{2}\right)^n \end{aligned} \tag{2}$$

It follows easily from (2) that the sequence c_n can be written in the following recursive form

$$\begin{aligned} &c_1 = 0 \quad , \quad c_2 = 10 \quad , \quad c_3 = 12 \\ &c_n = 5c_{n-2} + 4c_{n-3} \quad , \quad \forall n \geq 4 \end{aligned}$$

The proof is complete. $\square$