Problema U139. Find the least interval containing all values of the expression

$$E(x,y,z)=\frac{x}{x+2y}+\frac{y}{y+2z}+\frac{z}{z+2x}$$

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The expression $E(x,y,z)$ is unbounded above. Indeed setting up $x=n$, $y=\frac{1-n}{2}$, $z=0$ we have

$$E\left(n,\frac{1-n}{2},0\right)=\frac{n}{n+1-n}+1=n+1\to+\infty \text{ as } n\to\infty$$

Using the Cauchy-Schwartz inequality in Engel form we have

$$\begin{aligned}
E(x,y,z)&=\frac{x}{x+2y}+\frac{y}{y+2z}+\frac{z}{z+2x}=\\
&=\frac{x^2}{x^2+2xy}+\frac{y^2}{y^2+2yz}+\frac{z^2}{z^2+2xz}\geq\\
&\geq\frac{(x+y+z)^2}{x^2+y^2+z^2+2xy+2yz+2xz}=\\
&=\frac{(x+y+z)^2}{(x+y+z)^2}=1
\end{aligned}$$

and $E(x,y,z)$ attains the minimum value 1 as $x=y=z=1$.

Therefore least interval containing all values of $E(x,y,z)$ is $[1,+\infty)$. $\square$