

Problema U143. For a positive integer $n > 1$, determine

$$\lim_{x \rightarrow 0} \frac{\sin^2(x) \sin^2(nx)}{n^2 \sin^2(x) - \sin^2(nx)}$$

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First of all, let us rewrite the function in the following simpler form by using the well known identity $\sin^2 x = \frac{1-\cos x}{2}$:

$$\frac{\sin^2(x) \sin^2(nx)}{n^2 \sin^2(x) - \sin^2(nx)} = \frac{1 - \cos x - \cos(nx) + \cos x \cos(nx)}{2n^2 - 2n^2 \cos x - 2 + 2 \cos(nx)}$$

Now, a repeated application of L'Hospital's Rule gives the result

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1 - \cos x - \cos(nx) + \cos x \cos(nx)}{2n^2 - 2n^2 \cos x - 2 + 2 \cos(nx)} = \\ &= \lim_{x \rightarrow 0} \frac{\sin x + n \sin(nx) - \cos(nx) \sin x - n \cos x \sin(nx)}{2n^2 \sin x - 2n \sin(nx)} = \\ &= \lim_{x \rightarrow 0} \frac{\cos x + n^2 \cos(nx) - \cos x \cos(nx) - n^2 \cos x \cos(nx) + 2n \sin x \sin(nx)}{2n^2 \cos x - 2n^2 \cos(nx)} = \\ &= \lim_{x \rightarrow 0} \frac{-\sin x + (1 + 3n^2) \cos(nx) \sin x - n^3 \sin(nx) + (3n + n^3) \cos x \sin(nx)}{-2n^2 \sin x + 2n^3 \sin(nx)} = \\ &= \lim_{x \rightarrow 0} \frac{-\cos x + (1 + 6n^2 + n^4) \cos x \cos(nx) - n^4 \cos(nx) - 4(n + n^3) \sin x \sin(nx)}{-2n^2 \cos x + 2n^4 \cos(nx)} = \\ &= \frac{6n^2}{2n^4 - 2n^2} = \frac{3}{n^2 - 1} \end{aligned}$$

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