

**Problema U145.** Consider the determinant

$$D_n = \begin{vmatrix} 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^n & \cdots & n^n \end{vmatrix}$$

Find  $\lim_{n \rightarrow \infty} (D_n)^{\frac{1}{n^2 \log n}}$ .

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Taking into account the Vandermonde formula, we have

$$\begin{aligned} D_n &= \begin{vmatrix} 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^n & \cdots & n^n \end{vmatrix} = 2 \cdot 3 \cdots n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2^1 & \cdots & n^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-1} & \cdots & n^{n-1} \end{vmatrix} = \\ &= n! \prod_{i>j} (i-j) = n!(n-1)! \cdots 2!1! \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} (D_n)^{\frac{1}{n^2 \log n}} = \lim_{n \rightarrow \infty} e^{\frac{\log[n!(n-1)! \cdots 2!1!]}{n^2 \log n}}$$

In order to calculate the limit of the exponent  $\frac{\log[n!(n-1)! \cdots 2!1!]}{n^2 \log n}$ , we can use the theorem of Cesàro-Stolz:

Let  $(x_n)$  and  $(y_n)$  be two sequences of real numbers with  $(y_n)$  strictly positive, increasing, and unbounded. If

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L$$

then the limit  $\lim_{n \rightarrow \infty} \frac{x_n}{y_n}$  exists and is equal to  $L$ .

Now, calling  $x_n = \log[n!(n-1)! \cdots 2!1!]$  and  $y_n = n^2 \log n$ , we have that the sequence  $y_n$  is strictly positive, increasing and unbounded; furthermore

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} &= \lim_{n \rightarrow \infty} \frac{\log[(n+1)!n! \cdots 2!1!] - \log[n!(n-1)! \cdots 2!1!]}{(n+1)^2 \log(n+1) - n^2 \log n} \\ &= \lim_{n \rightarrow \infty} \frac{\log(n+1)!}{(n+1)^2 \log(n+1) - n^2 \log n} \\ &= \lim_{n \rightarrow \infty} \frac{\log(n+1) + \log n!}{n^2 (\log(n+1) - \log n) + (2n+1) \log(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \frac{\log n!}{n \log(n+1)}}{\frac{1}{\log(n+1)} \log\left(1 + \frac{1}{n}\right)^n + \frac{2n+1}{n}} = \frac{1}{2} \end{aligned}$$

where in the last step we have used the limits  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$  and

$$\lim_{n \rightarrow \infty} \frac{\log n!}{n \log(n+1)} = 1$$

The above limit also can be proved by means of Cesàro-Stolz theorem; in fact, setting  $u_n = \log n!$  and  $v_n = n \log(n+1)$  we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{u_{n+1} - u_n}{v_{n+1} - v_n} &= \lim_{n \rightarrow \infty} \frac{\log(n+1)! - \log n!}{(n+1) \log(n+2) - n \log(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{\log(n+1)}{n \log \frac{n+2}{n+1} + \log(n+2)} \\ &= \lim_{n \rightarrow \infty} \frac{\log(n+1)}{\log \left(1 + \frac{1}{n+1}\right)^n + \log(n+2)} = 1 \end{aligned}$$

Finally, the required limit is

$$\lim_{n \rightarrow \infty} (D_n)^{\frac{1}{n^2 \log n}} = \lim_{n \rightarrow \infty} e^{\frac{\log[n!(n-1)!\cdots 2!1!]}{n^2 \log n}} = e^{\frac{1}{2}} = \sqrt{e}$$

and we are done.  $\square$