

**Problema U146.** Let  $n$  be a positive integer. For all  $i, j = 1, \dots, n$  define  $S_n(i, j) = \sum_{k=1}^n k^{i+j}$ . Evaluate the determinant  $\Delta = |S_n(i, j)|$ .

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We have

$$|S_n(i, j)| = \begin{vmatrix} 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{vmatrix} \cdot \begin{vmatrix} 1 & 1^2 & \cdots & 1^n \\ 2^1 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \ddots & \vdots \\ n^1 & n^2 & \cdots & n^n \end{vmatrix}$$

From Vandermonde formula it follows that

$$\begin{aligned} \begin{vmatrix} 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{vmatrix} &= 2 \cdot 3 \cdots n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2^1 & \cdots & n^1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{n-1} & \cdots & n^{n-1} \end{vmatrix} = \\ &= n! \prod_{i>j} (i - j) = n!(n-1)! \cdots 2!1! \end{aligned}$$

Therefore

$$|S_n(i, j)| = (n!(n-1)! \cdots 2!1!)^2$$

and the desired result is established.  $\square$